TREE STRUCTURE FOR CONTRACTIBLE REGIONS IN $\mathbb{R}^3$

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Abstract. For contractible regions Ω in $\mathbb{R}^3$ with generic smooth boundary, we determine the global structure of the Blum medial axis M. We give an algorithm for decomposing M into "irreducible components" which are attached to each other along "fin curves". The attaching cannot be described by a tree structure as in the $2D$ case. However, a simplified but topologically equivalent medial structure $\hat{M}$ with the same irreducible components can be described by a two level tree structure. The top level describes the simplified form of the attaching, and the second level tree structure for each irreducible component specifies how to construct the component by attaching smooth medial sheets to the network of Y-branch curves. The conditions for these structures are complete in the sense that any region whose Blum medial axis satisfies the conditions is contractible.

Introduction

Suppose Ω is a bounded region in $\mathbb{R}^2$ or $\mathbb{R}^3$ with smooth generic boundary $\mathcal{B}$. The geometric structure of Ω is encoded by the Blum medial axis M [BN]. There is a considerable body of work devoted to computing the Blum medial axis in both 2 and 3 dimensions using a variety of methods including: the grassfire method (Kimia et al [KTZ]), the Hamilton–Jacobi skeleton (Siddiqi at al [SB]), and Voronoi methods (Szekely et al [SN]) among others (see also [P2]).

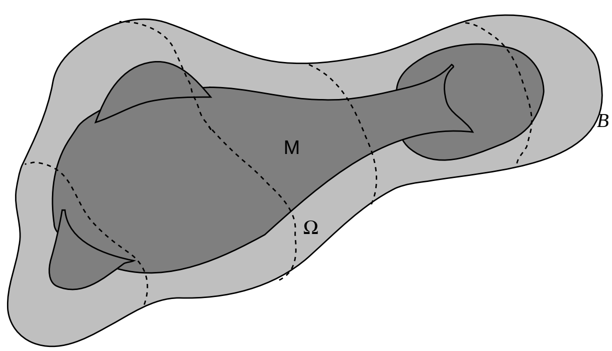

Figure 1. Blum medial axis for a Region in $\mathbb{R}^3$

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For example, in the case of a smooth "generic" boundary $\mathcal{B}$ of a region in $\mathbb{R}^2$, M is a collection of curves with Y-branching and end points, while for regions in $\mathbb{R}^3$, M is a two dimensional surface allowing special types of singular points. Such an M can be described as a "Whitney stratified set" whose local structure is given by a specific list of local models, given by Blum and Nagel for $n = 1$, by Yomdin [Y] and Mather [Ma] (for analogues of the Blum medial axis up through dimension ≤ 3, respectively ≤ 6), with a very explicit geometric description by Giblin [Gb] for $n = 2$ (see also §1). This exhibits M as a strong deformation retract of Ω, which means M and Ω have the same algebraic topological invariants even though they have different dimensions.

A basic question is how all of this information can be organized to represent the global geometric structure of M. For example, in $\mathbb{R}^2$, if the boundary $\mathcal{B}$ is a single closed curve, then the "Jordan Curve Theorem" and "Schoenflies Theorem" from topology together assert that Ω is topologically equivalent (homeomorphic) to the standard 2-disk. Such regions are called "contractible" because they can be shrunk down (contracted) within themselves to a point. Most outlines of objects have this property. An example of a noncontractible region would be the image of a donut, whose medial axis would have a curve encircling the hole.

For topological reasons, the Blum medial axis for a contractible region must have a tree structure. The vertices correspond to the Y−branch points and end points, and the edges, to the curve segments joining these points (see for example Fig. 2). Here by tree we mean a directed "unrooted tree", which is a directed graph without any cycles; although it need not have a single vertex from which other vertices can be reached following the directions of the edges. Importantly, matching of tree structures can be performed in polynomial time for certain classes of trees [Bi], [V], so there is an advantage to having such a tree structure on the Blum medial axis.

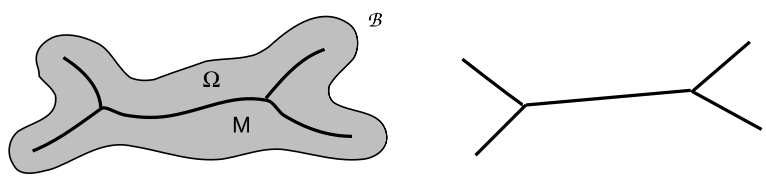

FIGURE 2. Blum medial axis for a Region in $\mathbb{R}^2$ with associated Tree structure.

In 3D, objects often are also contractible, which means that topologically they have no holes, as does a donut, nor enclosed cavities. Alternately, they can be viewed as being shaped from a lump of clay by reshaping, without introducing any holes or breaks and reattachings. Mads Nielsen asked whether again in the case of contractible $\Omega \subset \mathbb{R}^3$, the Blum medial axis has a tree structure, and what form does this tree structure take. If so, then this would be an essentiel step for developing of polynomial time matching algorithms for features of Blum medial axes.

By analogy with the $2D$-case, we might initially expect a simple description for the Blum medial axis M of Ω, where we replace curve segments of the $2D$–case with pieces of surfaces which are topologically 2–dimensional disks, and some of these 2–disks are attached along Y-branch curves ending in fin points as shown in Fig. 1. While such a Blum medial axis does correspond to a contractible region, it represents only a very small portion of the possibilities, and does not exhibit the intricate structure that is possible. We will explain in this paper just how complicated the structure can be.

As a point of comparison, in 2D it is possible to allow the boundary to deform using the flow defined by the curvature of the boundary. By the combined work of Gage, Hamilton, and Grayson [Ga], [GH], [Gr], the curvature flow will allow the boundary $\mathcal{B}$ to evolve so the fingers of the region shrink and the region ultimately simplifies and shrinks to a convex region which contracts to a point. The order of shrinking and disappearing of branches of M provides a natural ordering on the graph giving a tree structure. By contrast, if Ω is contractible in $\mathbb{R}^3$, then the analogous "mean curvature flow" may develop singularities, as for the example of the "dumbell" surface found by Grayson (see e.g. [Sn]). Thus, there is not a natural tree structure defined by mean curvature flow.

In this paper we provide a mixed answer to Nielsen's question. On the one hand we provide an algorithm for decomposing the medial axis into pieces which we call "irreducible medial components" M_i. which are attached to each other along "fin curves" (a simple example is given in fig. 1).

There are two crucial features which distinguish the general contractible case from Figure 1. First, unlike the simple model in fig. 1, the irreducible medial components can have a considerably more complicated structure; just how complicated will be explained by the general structure theorem we shall give. Second, each attaching of a irreducible component M_i along a fin curve may be to more than one other component; and different curves may give topologically inequivalent (i.e. nonhomeomorphic) medial axes. Hence, to recover the complete structure of M from the M_i requires more than just a graph describing the attachings.

The goal then is twofold: to give a concise description of the structure of each irreducible medial component which captures its topological structure; and second, to provide an algorithm for reconstructing M from the M_i by attaching along fin curves. We shall also see that not all of the irreducible components are uniquely defined because the presence of certain types of fin curves requires us to make choices, leading to different representations. For essentially equivalent medial axes, different geometric conditions make different choices seem "more natural". Also, the attaching process is actually an inductive process and requires considerably more data than is needed for the $2D$–case. For a complete description, Nielsen's question has a negative answer, although we provide a substitute description in place of a graph.

However, there is a simplified version of M that sacrifices some of the detail (yielding a medial structure $\hat{M}$ equivalent to M in a weaker topological sense of "homotopy equivalence"). For this version, we retain the structure of the irreducible components, but use a simplified form of attaching, which can be described by a graph $\Gamma(M)$. If Ω is contractible, then this graph turns out to be tree. Second, we describe the structure of each irreducible medial component M_i by a secondary graph $\Lambda(M_i)$. This graph describes how the smooth surface sheets are attached to the network of Y–branch curves. In the contractible case each $\Lambda(M_i)$ will also be a tree. Finally, if Ω and (hence) M are contractible, then each M_i must also be contractible. However, it is somewhat counterintuitive that the smooth surface sheets which make up the M_i need not be contractible; nor must the connected components of the network of Y–branch curves form trees. Exactly how complicated the M_i can be and how they are required to be attached to the Y–network curves is part of the data attached to the secondary graph.

The complete characterization of this data for the contractible case will follow from a general structure theorem for the Blum medial axis for arbitrary compact regions in $\mathbb{R}^3$ [D3]. When this theorem is applied to contractible regions, it yields a multilevel directed tree structure which we describe in this paper.

Before beginning with the detailed description of the structure, we give a very brief overview.

Description of the Structure for Contractible Regions

The structure we give has several levels. First, there is the *top level directed tree structure* $\Gamma(M)$. We decompose M into "irreducible medial components" M_i. These correspond to the vertices of $\Gamma(M)$. Such components do not have any fin points (nor "fin curves" joining them). A directed edge from M_i to M_j corresponds to the attaching of M_i to M_j along a single created "fin curve" joining two fin points. The resulting space obtained by attaching the M_i as indicated by the edges of $\Gamma(M)$ is a simplified form $\hat{M}$ of the original M. M can be recovered from $\hat{M}$ by certain sliding operations along fin curves according to additional data defined from M.

If we think of associating to each M_i a region Ω_i with boundary $\mathcal{B}_i$, then the graph $\Gamma(M)$ also describes how to recover the initial region and boundary $(\Omega, \mathcal{B})$ from the individual $(\Omega_i, \mathcal{B}_i)$ by taking "connected sums" (see Fig. 14).

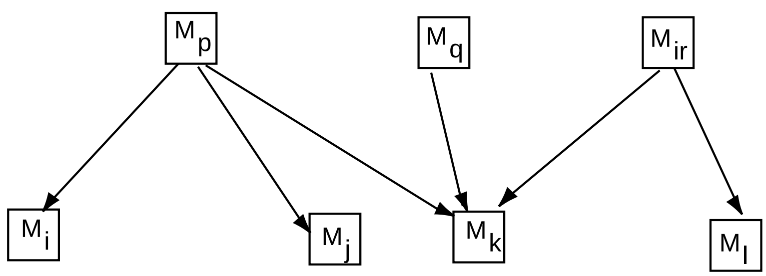

FIGURE 3. Tree structure given by decomposition into irreducible medial components with edges indicating the attaching along "fin curves"

If Ω (and hence M) is contractible, then $\Gamma(M)$ must be a tree as in Fig. 3 (see Corollary 2.5). If M is contractible, the irreducible medial components M_i must again be contractible.

Then, the structure of each irreducible medial component M_i is described by a *second level directed tree structure* $\Lambda(M_i)$. This tree structure has two types of vertices: S–vertices and Y–nodes. Also, an edge of $\Lambda(M_i)$ can only go from an S–vertex to a Y–node. The nodes correspond to connected components $\mathcal{Y}_{ij}$ of the "Y–network" $\mathcal{Y}_i$ of M_i. The "Y–network" of M_i is the collection of Y–branch curves together with vertices which are the "6–junction points" where 6 sheets of the Blum medial axis come together in a point. The S–vertices correspond to the connected smooth sheets S_{ij} of the component M_i. In fact, because M_i is contractible, we shall see that each S_{ij} must be topologically a 2–disk with a finite number of holes. An edge from an S–vertex representing S_{ij} to a Y–node for $\mathcal{Y}_{ik}$ represents the attaching of a boundary circle of S_{ij} to the component $\mathcal{Y}_{ik}$, as in Fig 4.

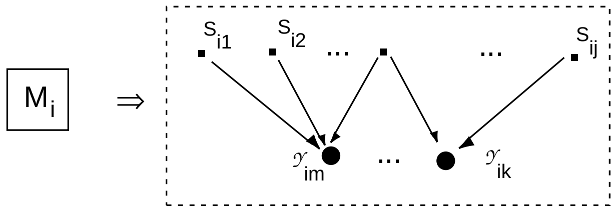

FIGURE 4. Directed Tree structure for contractible irreducible medial component M_i: S–vertices ■ representing connected medial sheets S_{ij} joined to Y–nodes • representing components $\mathcal{Y}_{ik}$ of the "Y–network"

Finally, at the third level, we assign data to the S–vertices, the Y–nodes, and the edges. To an S–vertex we assign a pair (h, e) with h denoting the number of holes in S_{ij}, and $e = 0, 1$ depending on whether there are no or one boundary circle of S_{ij} which is also an edge curve of the component M_i. Again the topology prevents there being more than one. To each Y–node, we assign a "4-valent extended graph" Π_{ik} which gives the structure of the Y–network component $\mathcal{Y}_{ik}$ associated to that node. Finally, to each edge we assign attaching data which describes how topologically the boundary circles are attached to the Y–network $\mathcal{Y}_{ik}$.

Along with these graphs and data are relations which must be satisfied by the various numerical data associated to the vertices, edges ,etc. We shall also give these relations in §3. These relations are necessary, but they are also sufficent to ensure that the resulting Blum medial axis reconstructed according to the data is contractible.

The structure described here is a consequence of a general structure theorem. The full details of that theorem and its proof requires considerable use of algebraic topology and will not be presented here, see [D3]. A reader wishing to become familiar with some of the ideas and terminology of algebraic topology to better understand this paper, is referred to the book [Mu] or the more elementary [Go].

The author is especially grateful to Mads Nielsen for initially raising the question about the global structure of the medial axis and the possibility of a tree structure in the contractible case.

CONTENTS

1. Generic Local Structure of Blum Medial Axis

We consider a region $\Omega \subset \mathbb{R}^3$ with generic smooth boundary $\mathcal{B}$. We recall [BN] that the Blum medial axis M is the locus of centers of spheres in $\mathbb{R}^3$ which are contained in the region Ω and are tangent to the boundary $\mathcal{B}$ in at least two points (or having a single degenerate tangency). This locus has also been called the "central set"in mathematics literature, see Yomdin [Y]. It can alternately be described, as in Mather [Ma] as the Maxwell set for the family of distance functions on the boundary.

The family of distance functions on $\mathcal{B}$ is the parametrized family $f(x, u) : \mathcal{B} \times \Omega_0 \to \mathbb{R}$ on $\mathcal{B}$ for parameters $u \in \Omega_0$, the interior of Ω. It is given by $f(x, u) = \|x - u\|$. The "Maxwell set"for this parametrized family consists of those parameter values u such that $f(\cdot, u)$ has two or more points with the same minimum value. This is exactly the Blum medial axis; and because of its alternate description as the Maxwell set for the family of distance functions, a general result of Mather [Ma], implies that it exhibits generic properties given from singularity theory. In particular, for generic $\mathcal{B}$, it is a 2–dimensional Whitney stratified set. This means it can be decomposed into pieces which are surfaces, curves or points, in such a way that the local structure looks the same along the curves. At the special points, the set looks like a collection of curves and surface pieces approaching the point in a very regular way. We do not try to be more precise because there is a further explicit description of local models for the medial axis. As already mentioned, by the work of Yomdin [Y], Mather [Ma], and more recently Giblin who gave a very explicit geometric description [Gb], the local models for singular points in the generic case have one of the following four local forms in Fig. 5 (where we include the radial vector fields from M to the points of tangency on the boundary). These local models result from the classification of the local properties of family of distance functions, up to differentiable change of coordinates. See also the recent book [PS].

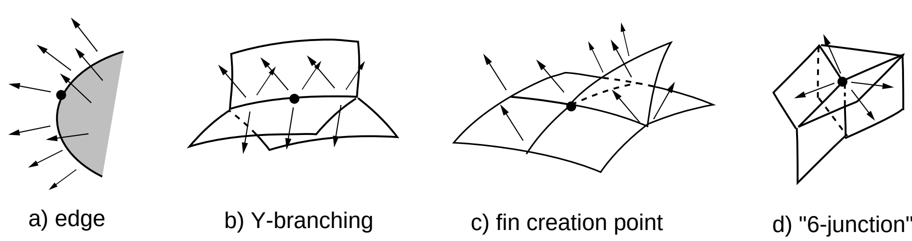

Figure 5. Four types of local generic structure for Blum Medial axes in $\mathbb{R}^3$ (other than smooth surface points) and the associated Radial Vector Fields. For each type, a point of that type is darkened.

Then, M consists of the following: i) 2D smooth connected sheets, which we refer to as *medial sheets*, ii) Y–junction curves along which three sheets meet in a Y-branching pattern; iii) edge curves consisting of edge points of M; iv) fin points; and v) 6–junction points where six medial sheets meet along with 4 Y–junction curves. Thus, connected components of Y–junction curves end either at fin points or 6–junction points; while edge curves only end at fin points.

We initially refer to the union of Y–junction curves, fin points, and 6–junction points as the *Y–network* $\mathcal{Y}$. Shortly we shall simplify the Y–network by removing curves ending at fin points, and still refer to the simplified network as the Y–network.

Attaching Medial Sheets to the Y–network. The basic view of the medial axis is to view it as being built up by first forming a network of curves meeting at vertices, and then attaching the smooth medial sheets to the curve network. To do so we must first be able to separate the medial sheets from the Y–branch curves, and fin and 6–junction points. We do this by "cutting M along $\mathcal{Y}$" in order to separate the sheets. There is a formal mathematical way to do this which corresponds to adding closure points to the smooth sheets (it is possible that more than one part of a sheet may have the same point in $\mathcal{Y}$ as a closure point, in which case we add one distinct closure point for each part of the sheet). The resulting objects are smooth sheets but now with piecewise smooth boundaries, which consist of the points from $\mathcal{Y}$ and the edge curves of the medial axis. As such they are topologically compact surfaces with boundaries. An example of this is given in Fig. 6.

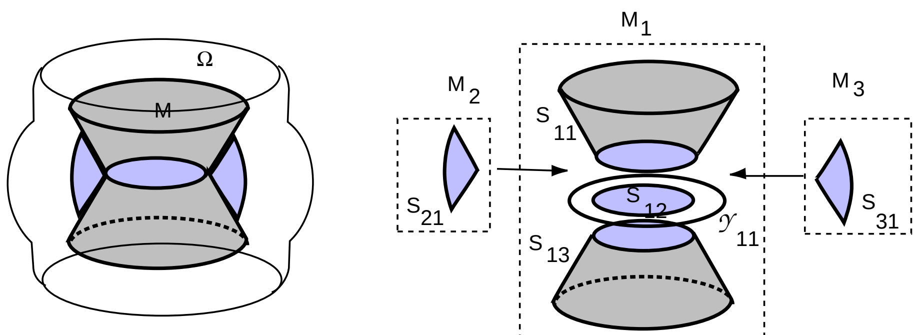

FIGURE 6. Cutting the Medial Axis first along fin curves and then along the Y–network $\mathcal{Y}$

We refer to the closed sheets as *medial sheets*. Their boundaries consist of a finite union of (homeomorphic copies of) circles. Then, M is obtained by attaching the boundary circles of closed sheets S_i to $\mathcal{Y}$, which is a network of curves meeting at 6–junction points or ending at fin points. Actually if we carry this out without any preliminary analysis, we end up making unecessary cuts in certain closed medial sheets because of the presence of fin points. Thus, we first turn to how we can eliminate fin curves joining fin points before we perform the cuts to obtain the medial sheets.

2. Tree Structure by Decomposition into Irreducible Medial Components

We decompose the medial axis into irreducible medial components which are attached to each other along fin curves.

Separating the Medial Axis along Fin Curves

Suppose we have on the Blum medial axis a fin point p. Then near p, we can distinguish the "fin sheet" which is the sheet along the Y–branch curve that contains a medial edge curve near p. We can begin following the Y–branch curve from p, while keeping track of the fin sheet close to the curve. This sheet can be identified as the connected continuation of the fin sheet near the Y–branch curve as we move along the curve. Eventually one of two things must happen: either we reach a 6–junction point or another fin point. First, if the Y-branch curve meets a 6–junction point, then we continue the fin sheet so it is path–connected near the 6–junction point. We can follow the Y–branch curve which the sheet meets as it continues through the 6–junction point. After the 6–junction point we have identified both the corresponding continuation of the Y–branch curve, and the fin sheet close to the continuing Y–branch curve. We can do this for each 6–junction point it encounters. As M is compact, eventually the Y–branch curve must meet another fin point. We shall refer to this Y–branch curve from one fin point to the other as a *fin curve*.

At the end of the fin curve, what was identified as the fin sheet (close to the Y–branch curve) from the beginning may or may not be the fin sheet for the end fin point. If this same sheet is a fin sheet at both ends, then we refer to the fin curve as being "essential". If instead it is only a fin sheet at one end, but not the other, then we refer to the corresponding fin curve as "inessential"(later discussion will explain the reason for these labels). Examples of these are shown in Fig. 7.

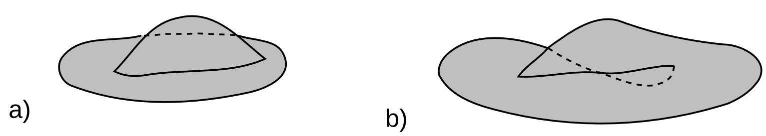

FIGURE 7. Two possibilities for fin curves on a medial sheet: a) essential fin curve b) inessential fin curve

An example of a region with just an inessential fin curve is given in Fig. 8 and might be called a "Mobius board", a surf board but with a "Mobius band"twist.

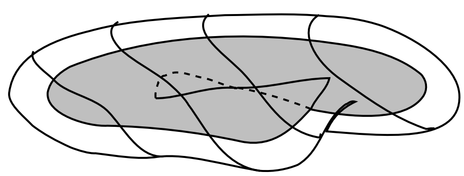

FIGURE 8. "Mobius Board"with an "inessential fin curve"

We next turn to the task of cutting along the fin curves. First, we can cut the fin sheet along a fin curve. Then, we can take the two remaining sheets still attached along that fin curve (which meet at a positive angle) and smooth them to form a smooth sheet along the curve, with former 6–junction points.on the fin curve becoming Y–branch points (for another Y–branch curve). The result depends upon a further distinction for essential fin curves. A *type*-1 *essential fin curve* will be one which only intersects other essential fin curves at 6–junction points; otherwise, it shares a segment of Y–branch curve with another essential fin curve, and it will be *type*-2 *essential fin curve* (see e.g. Fig. 9). If we cut along a type-1 essential fin curve, then the fin sheet becomes disconnected from the other sheets (at least along the curve) and this does not alter any other essential fin curve. If we cut along a type-2 essential fin curve, then it will alter the structure of the other essential fin curves sharing a segment of Y–branch curve with it. Hence, we can follow the algorithm.

Algorithm for Decomposing Medial Axis into Irreducible Components

(1) Identify all type-1 essential fin curves and systematically cut along these type-1 essential fin curves (it does not matter which order we choose).
(2) After cutting along all type-1 essential fin curves, we may change certain inessential fin curves to type-1 essential ones. If so return to step 1).
(3) There only remain type-2 essential fin curves and inessential fin curves. Choose an essential fin curve and cut along it. If a type-1 essential fin curve is created, return to step 1). Otherwise, repeat this step until no essential fin curves remain.
(4) When there are no other essential fin curves, choose an inessential fin curve which crosses a 6–junction point, and cut it from one side until we cut across one 6–junction point.
(5) Check whether we have created an essential fin curve. If so then we cut along it, and repeat the earlier steps 1- 3).
(6) If no essential fin curve is created, then we repeat step 3) until there are only inessential fin curves which do not cross 6–junction points.
(7) Finally we can contract each such remaining inessential fin curve to a point, producing part of a smooth sheet (i.e. in effect the fin curve disappears).
(8) The remaining connected pieces are the "irreducible medial components" M_i of M.

Remark 2.1. The distinct connected pieces created following steps 1) and 2) are intrinsic to M; while those created using steps 3) and 4) are not because choices are involved. Which choices are made typically depends on the given situation and the importance we subjectively assign to how sheets are attached.

Example 2.2. In a) of fig. 9, we have a contractible medial axis with a pair of type-2 essential fin curves. Depending on which essential fin curve we choose, $1-2$ or $3-4$, we choose to cut along in step 3), we obtain either b) or c), which leads to different attachings (and hence top level graph) for the irreducible medial components. An alternate possibility would be to cut each fin sheet along the fin curves and view them as being attached partially along the edge of a fourth sheet. Again, the exact geometric form of M may suggest one choice being preferred over the others.

Example 2.3. In a) of fig. 10, we have a contractible medial axis with 10 fin points 1-10, and all fin curves are inessential. Depending on how we choose cuts in step 4) of the algorithm, we can end up with 1, 2, or 3 irreducible medial components.

If we cut from 4 through the first 6–junction point, then 3-6 becomes an essential fin curve, and we cut away the fin sheet M_1 as in b) of fig. 10. Then further cutting from 8 through the first 6–junction point, we create another essential fin curve 2-9. Cutting along it creates a second fin sheet M_2. The remaining inessential fin curves 1-4, 5-7, and 8-10 can be contracted to points on edges of the third sheet M_3. Each of these 3 medial sheets are then irreducible components.

Alternatively, after the first cut, we could have instead cut from 9, and then from 4 again, and then only inessential fin curves remain without 6–junction points, so

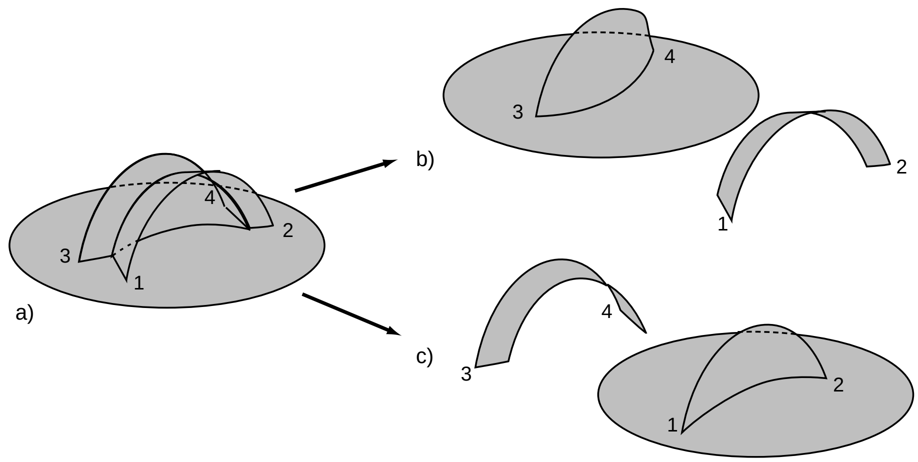

FIGURE 9. Nonuniqueness of medial decomposition resulting from type-2 essential fin curves: a) is a contractible medial axis with only type-2 essential fin curves; and b) and c) illustrate the results from cutting along the fin curves $1-2$ or $3-4$.

they contract to a second sheet, and we only obtain two irreducible components. Thirdly, we could have begun cutting from 7, then 8, and then 4 twice and we would obtain only a single medial sheet with inessential fin curves, leading to a single irreducible component.

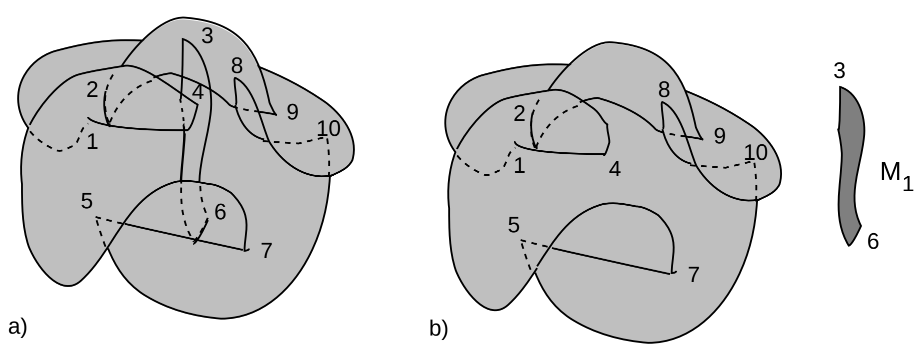

FIGURE 10. Nonuniqueness of medial decomposition resulting from inessential fin curves: a) is a contractible medial axis with only inessential fin curves; and b) illustrates the cutting of irreducible medial component M_1 after cutting from fin point 4.

To reverse the algorithm and reconstruct M from the M_i requires that: we first create the appropriate inessential fin curves from appropriate edges of the M_i and then reverse the steps by attaching the M_i along edges to fin curves which can cross multiple components. The attaching is an inductive process which for a given component requires the list of sucessive components and the embedded curves in each component along which the attaching will occur.

Top Level Tree Structure. There is a simplified version of the attaching of the M_i which sacrifices some of the detailed structure of M. The resulting simplified medial $\hat{M}$ structure still recovers M up to a weaker topological equivalence (homotopy equivalence). The simplified attaching is described by a graph structure $\Gamma(M)$. To obtain this graph, instead of cutting along fin curves, we alternately slide the sheets along the fin curves.

Sliding along Fin Curves. For a single fin curve γ from fin points p to q, we have the fin sheet S for p. Instead of cutting along the fin curve γ beginning at p, we alternately slide the sheet along the fin curve. Specifically, we deform γ to γ_t whose initial point progressively moves along γ passing all of the 6–junction points. Now γ becomes a fin curve γ' which no longer passes through any 6–junction points (the former 6–junction points are now just points on a Y–branch curve (as in Fig 11)).

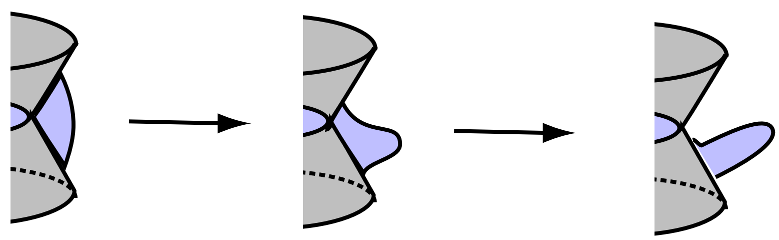

FIGURE 11. Deforming a fin curve so it misses 6–junction points

The sheet is then still attached along the fin curve, but only to a single smooth sheet (and without 6–junction points). After sliding the sheet, we can smooth points along the remaining Y–branch curve where the remaining two sheets are attached to obtain locally a smooth surface.

Then, by a result from topology, attaching the sheet using γ' instead of γ gives spaces which are homotopy equivalent (which is a weaker form of topological equivalence which still implies that the algebraic-topological invariants agree). There are two possiblilites .

Lemma 2.4. *In the preceding situation, let γ' be a fin curve which does not meet a 6–junction point. There are two possibilities.*

1. *γ is an essential fin curve. Then, when we cut the sheet along the fin curve γ', the sheet locally becomes disconnected from the remaining sheets of the fin curve (as in a) of Fig. 7).*
2. *Instead γ is an inessential fin curve. Then, we may shrink γ' to a point on a medial edge, and the attached sheet becomes part of the sheets it was attached to (as in b) of Fig. 7).*

This Lemma says that an "inessential fin curve"can be eliminated without any change in topology (i.e. homotopy type).

Remark . If we slide along the fin curves and then cut, we obtain the same components as we would have had by just cutting as originally described.

Now beginning with a Blum medial axis, we may slide sheets along fin curves and contract inessential fin curves. This gives rise to a "top level directed graph structure"$\Gamma(M)$ which is a graph with vertices for each irreducible component M_i, and directed edges from the vertex M_i to M_j for each edge of M_i attached to M_j along an (essential) fin curve. Then by Corollary 2.6 of [D3], there is the following relation between the top level graph $\Gamma(M)$, the irreducible medial components $\{M_i\}_{i=1}^r$ and the full medial axis M.

Corollary 2.5. *If M is contractible, then so is each M_i contractible; and furthermore, $\Gamma(M)$ is a (directed) tree.*

In the special case that Ω is contractible, then so is M (as it is a strong deformation retract of Ω). Thus, Corollary 2.5 gives the top level directed tree structure $\Gamma(M)$.

Remark 2.6. We have already discussed the nonuniqueness of the decomposition into irreducible medial components. This also extends to $\Gamma(M)$ because there is not a unique choice of irreducible component onto which we slide a fin curve. Thus, there is for each directed edge, a list of irreducible components that it could be attached to. This is illustrated in fig. 12, where the contractible medial axis in a) has medial sheets M_1, M_2, M_3 and M_4 attached to all three. The graph of all attachings is shown in a) of fig. 13. This graph does not completely describe the structure of M. If we had used instead a fin curve which stays on one side of the two holes, then we would obtain a medial axis not homeomorphic to the M shown in a) of fig. 12.

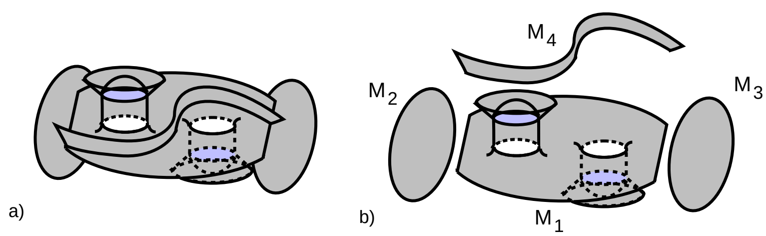

FIGURE 12. a) A contractible medial axis, with attaching of component M_4 to multiple components M_1, M_2, M_3; b) Irreducible medial components for a).

Two different ways to slide the sheets yielding the trees are shown in b) and c) of fig. 13. Thus, there are "equivalent"directed tree structures corresponding to the same Blum medial axis but using different choices of possible attachings. However, independent of which choices we make, the resulting top level graph will always be a tree.

Remark 2.7 (Geometric Criteria for Uniqueness)**.** For a specific class of objects, it might be possible to assign other external criteria, such as *importance of medial sheets*, for making preferred choices at those steps of the algorithm where there is nonuniqueness. One type of measure of importance is geometric. For example, following the image of a medial sheet M_i under the radial flow defined in [D1] from M to the boundary $\mathcal{B}$ of the region Ω defines both subregions Ω_i of Ω and $\mathcal{B}_i$ of $\mathcal{B}$. Hence, the volume of Ω_i or the surface area of $\mathcal{B}_i$ provide geometric measures of significance for the sheets. Furthermore, in addition to the medial axis M, we also have the multivalued radial vector field U on M from points on M to the points of tangency in $\mathcal{B}$. Then, by results in [D4], we can compute the volumes and boundary areas as medial integrals on M, with respect to a medial measure, of expressions obtained from the radial geometry on M. Hence, these types of measures of geometric significance are determined completely from (M, U).

Then, we can use such measures of relative importance to decide which choices to make in the algorithm. For example, in step 3) we could choose the fin curve with fin sheet of minimal significance, and similarly in step 4). Lastly, in deciding how to slide along fin curves to a medial sheet to define the top level graph, we could slide to the sheet with maximum significance. In fig. 12, such a criterion would lead to the choice of top level graph b) in fig. 13.

Of course, our comments here are only to indicate possibilities, rather than give a detailed prescription for carrying out these ideas at this time.

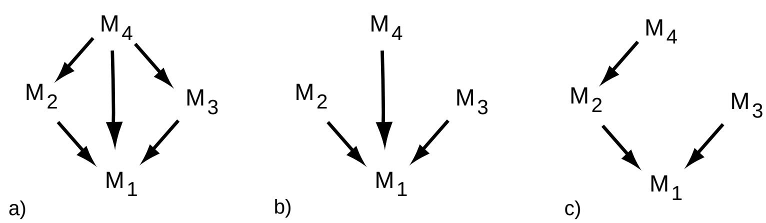

FIGURE 13. a) The graph of all attachings for the medial axis in fig. 12; b) and c) represent different ways to slide the sheets to yield a top level tree.

Structure of $(\Omega, \mathcal{B})$ *as a Boundary Connected Sum.* There are three operations involved in this process: movement of fin curves so they do not pass through 6–junction points; the separation of a region along essential fin curves; and the contraction of inessential fin curves. The first and third of these operations arise as generic transitions from local deformations of the region's shape. The second requires a change in region topology. There has been work by Giblin and Kimia [GK], building on the results of Bogaevski [Bg], [Bg2], which determines the generic local transitions occurring in one parameter deformations of regions. Their results raise the question of whether the first and third operations can be realized from global deformations of the region's shape? If so then there is also an interpretation of the second operation as a geometric operation of boundary connected sum.

Given two surfaces X_1 and X_2, the connected sum $X_1 \sharp X_2$ is obtained by cutting out a disk from each and gluing them together along the edges of the disks. If The X_i are boundaries of 3D regions G_i then the same construction gives the *boundary connected sum* of the G_i with boundary $X_1 \sharp X_2$. Given one of the M_i, we can use the radial vector field defined everywhere on M_i except in neighborhoods of the fin curves. Following the radial flow on this subset (as defined in [D1] or see [D3]) gives part of Ω. From the boundaries of the neighborhoods, we obtain under the radial flow cylinders homeomorphic to $S^1 \times [0, 1]$. If we topologically fill in the cylinder to make it solid, we obtain a region Ω_i with boundary $\mathcal{B}_i$. We can then obtain the boundary connected sum of the Ω_i using the graph $\Gamma(M)$ to determine which components will be attached to each other. We obtain a region which is homeomorphic to Ω, with boundary, which is the connected sum of the

$\mathcal{B}_i$, is homeomorphic to $\mathcal{B}$. Fig. 14 represents the region in Fig. 1 as a boundary connected sum of regions.

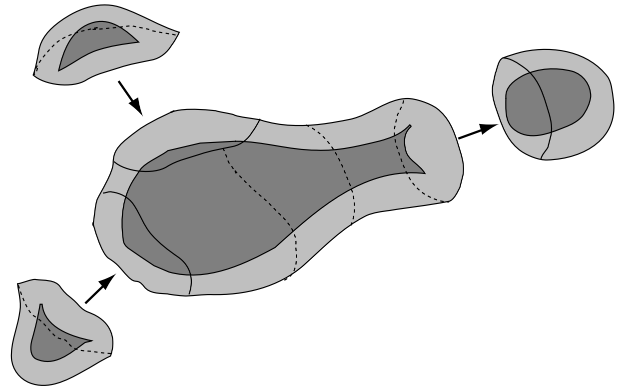

FIGURE 14. Boundary Connected Sum of 3-dimensional Regions

3. Basic Structure of Irreducible Medial Components

The previous section reduces understanding the structure of the Blum medial axis M to the case of the individual irreducible medial components M_i. These medial components are connected and locally have only Y-junction points, edge points, and 6-junction points (but no fin points). We also note that the network of Y–branched curves has been altered by the removal of the fin curves. We denote the new collection of Y–branched curves together with any remaining 6–junction points by $\mathcal{Y}$ and refer to it as the *Y–network.* Then, $\mathcal{Y} = \cup_i \mathcal{Y}_i$, where each $\mathcal{Y}_i$ is the Y–network for M_i.

After we have separated these medial components, we may then cut along the Y–network curves to represent the medial component as obtained from the medial sheets by attaching them to the Y–network. However, we note that the base sheets for fin points will not be cut along the removed fin curve.

There are three steps for analyzing the construction of the irreducible medial components from the medial sheets S_{ij} and Y–network $\mathcal{Y}_i$: i) determining the structure of each medial sheet; ii) analyzing the structure of the components $\mathcal{Y}_{ik}$ of $\mathcal{Y}_i$; and iii) determining the topological properties of the attaching maps of S_{ij} to $\mathcal{Y}_{ik}$.

Structure of the Medial Sheets and Y–Network. In the case that M, and hence the M_i, are each contractible, the medial sheets have the following special form.

Proposition 3.1. *If M_i is contractible, then each medial sheet S_{ij} is topologically equivalent (homeomorphic) to a 2–disk with a finite number of holes. Also, at most one boundary circle (of a hole or the 2-disk itself) will be an edge curve of the medial axis (corresponding to a crest curve on $\mathcal{B}$) and the rest will be attached to $\mathcal{Y}_i$.*

We emphasize that such a medial sheet, although restricted by topology, may take several apparently distinct geometric forms, as e.g. Fig. 15.

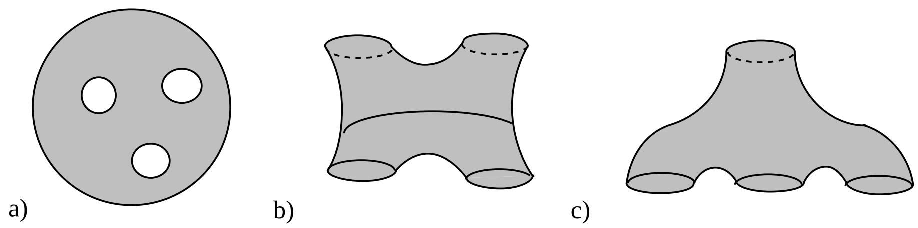

FIGURE 15. Surface with Boundary a) and several different geometric forms b) and c).

Second, we also want to precisely characterize the structure of each connected component $\mathcal{Y}_{ij}$ of $\mathcal{Y}_i$, the Y–network of M_i. We can view each $\mathcal{Y}_{ij}$ as being characterized by an "extended graph" Π_{ij}, where we have vertices corresponding to 6–junction points, and for each curve in $\mathcal{Y}_{ij}$ between two 6–junction points we associate an edge between the corresponding vertices. Since there may be more than one edge between two vertices, instead of having a graph we will refer to it as an "extended graph". In fact there can be an edge from a vertex to itself. However, by the properties of 6–junction points, there will be a total of 4 edges ending at each vertex (where an edge with just one vertex is counted twice). We refer to Π_{ij} as a 4*–valent extended graph*. We also allow for two "degenerate possibilities". The first is that the graph is empty, and we denote it by $\emptyset$. The second possibility is the extended graph with 0 vertices - it consists of a single closed curve and will be denote by $\circ$.

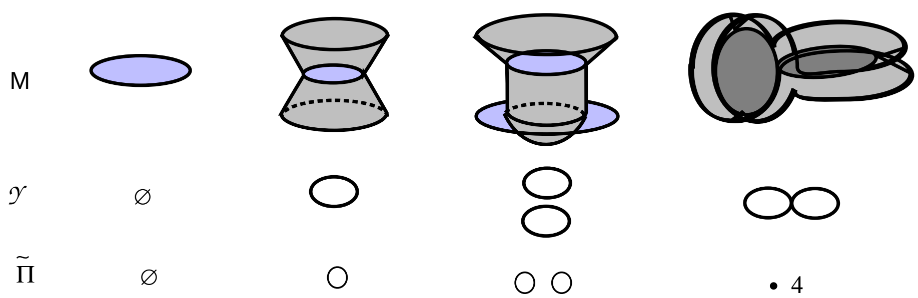

FIGURE 16. Examples of Irreducible Medial Components M , their Y–Networks $\mathcal{Y}$, and the Reduced Network Graphs $\tilde{\Pi}$ (see below).

The only invariant of the topology (i.e. homotopy type) of such an extended graph is its fundamental group. This can be determined as follows.

Proposition 3.2. *If a nonempty connected 4–valent extended graph* Π *has* k *vertices, then the fundamental group* $\pi_1(\Pi)$ *is a free group on* $k+1$ *generators.*

Remark 3.3. This has as a consequence that for a connected component $\mathcal{Y}_{ij}$ with v_{ij} vertices, the maximum number of independent cycles is $v_{ij}+1$. If $\mathcal{Y}_i$ has a total of c_i connected components and $v_i = \sum_j v_{ij}$, then $\mathcal{Y}_i$ has a maximum of $v_i + c_i$

independent cycles in its graph. In the examples in Fig. 16, $\mathcal{Y}$ has respectively 0, 1, 2, 2 as the maximum number of independent cycles.

Second Level Directed Tree associated to a Medial Component. We then introduce for each irreducible medial component M_i a graph $\Lambda(M_i)$ with two types of vertices: S-vertices corresponding to the medial sheets S_{ik} of M_i, and Y–nodes which correspond to connected components $\mathcal{Y}_{ik}$ of the Y–network $\mathcal{Y}_i$ of M_i. Finally we have an edge from S_{ik} to $\mathcal{Y}_{ik}$ if a boundary circle of S_{ik} is attached to $\mathcal{Y}_{ik}$.

Proposition 3.4. *If M_i is contractible, then $\Lambda(M_i)$ is a directed tree. Moreover, if there is an edge joining the S–vertex S_{ij} to the Y–node $\mathcal{Y}_{ik}$, then only a single boundary circle of the medial sheet S_{ij} is attached to a component $\mathcal{Y}_{ik}$.*

We briefly explain what we mean by topological data for attaching a boundary circle to a component $\mathcal{Y}_{ik}$. If we assign a direction and a label to each edge of $\mathcal{Y}_{ik}$ as in Fig. 17, then the topological data is a "word"such as $a_1a_2a_5a_6a_{12}^{-1}a_{11}^{-1}a_8^{-1}a_7$ where an exponent of -1 indicates crossing the segment in the direction opposite the arrow. For this particular word, a circle with given orientation and distinguished point, would send the point to the beginning of a_1 and then follow the path as indicated ending again at the beginning point of a_1. It would trace out a figure–8 type curve (with extra loop) in this case.

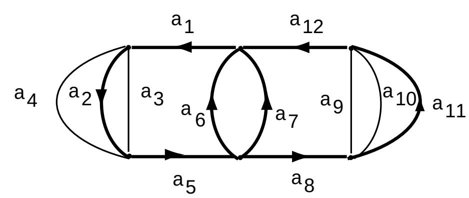

FIGURE 17. Attaching boundary circle to Y–Network according to the "word"$a_1a_2a_5a_6a_{12}^{-1}a_{11}^{-1}a_8^{-1}a_7$ indicated by darkened edges

Reduced Network Graph $\tilde{\Pi}_{ik}$

In order to work with such graphs it is easiest to simplify the graph Π_{ik} as follows. If a vertex has a loop, which is an edge which begins and ends at the vertex, then we suppress the edge and instead assign a number to the vertex indicating how many edges to the vertex come from such loops (it is 0, 2 or 4). Also, if there is more than one edge from a vertex to another vertex, we instead use a single graph edge between them with a number attached indicating the number of edges (again between 1 and 4). If for the vertex the number of loops is 0 then we suppress it; likewise if there is only 1 edge between two vertices, we usually suppress the 1. We refer to this graph $\tilde{\Pi}_{ik}$ as the *reduced Y–network graph* of $\mathcal{Y}_{ik}$. For such a reduced graph, at a vertex the sum of the vertex number and edge numbers always equals 4. There are very few local possibilities for such reduced graphs about vertices. These are all shown in Fig. 18

Third Level structure and Relations. Finally we define the *third level of the structure* by data associated to the tree. The data attached to a S–vertex is (h, e), where h denotes the the number of boundary circles and $e = 0, 1$ counting the number of medial edge curves on S_{ij}. The data attached to the node $\mathcal{Y}_{ik}$ is the

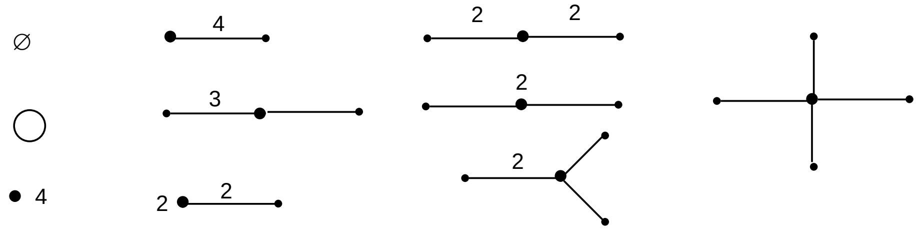

FIGURE 18. The Possible Local Vertex Structures (at enlarged vertex) for reduced Y–network graphs

4–valent directed graph Π_{ik} (or $\tilde{\Pi}_{ik}$). The data on an edge from a vertex S_{ij} to a node $\mathcal{Y}_{ik}$ is the attaching data of the boundary circle to $\mathcal{Y}_{ik}$.

We further establish two basic relations for an irreducible component M_i. We let: s_i denote the number of connected sheets of M_i; v_{ik}, the number of 6–junction points in the component $\mathcal{Y}_{ik}$, with $v_i = \Sigma v_{ik}$, the number of 6–junction points in the total Y–network $\mathcal{Y}_i$; e_i the number of edge curves in M_i; and c_i the number of connected components $\mathcal{Y}_{ik}$ of the Y–network $\mathcal{Y}_i$.

Proposition 3.5 (Euler Relation)**.** *If M_i is a contractible irreducible medial component, then with the preceding notation,*

$$s_i \; - \; e_i \quad = \quad v_i \; + \; c_i \tag{3.1}$$

In this equation, the LHS is an expression for the number of sheets without a medial axis edge curve. In the contractible case this must equal the RHS which is Y–network data, the sum of the number of 6–junction points and number of connected components in the Y–network (by an earlier remark, this sum is the maximum number of independent cycles in the Y–network). We see in Example 4.3, this Euler relation may fail for a noncontractible region.

The second basic relation must be stated in terms of the fundamental group.

Fundamental Group Condition

There is one further condition needed to characterize contractible regions in terms of the Blum medial axis. This can only be stated in terms of the Fundamental groups of the irreducible medial components.

We consider a single irreducible medial component M_i. First, for each Y–network component $\mathcal{Y}_{ik}$, we can choose a maximal tree T_k of curves in $\mathcal{Y}_{ik}$ (choose a maximal tree $\tilde{T}_k$ in the graph Π_{ik}, and then choose a Y-branch curve in $\mathcal{Y}_{ik}$ corresponding to each edge in $\tilde{T}_k$). Second, each medial sheet S_{ij} is topologically a 2–disk with a finite number of holes. We construct from a point z_{j0} in the interior of S_{ij}, nonintersecting paths to points $x_{j\ell}$ on the boundary circles $C_{j\ell}$, where we may suppose the circles are arranged in increasing order as we turn in a counterclockwise direction from z_{j0}, with the first circle being the boundary of the disk. We can choose the $x_{j\ell}$ so that if $C_{j\ell}$ is not an edge curve of M_i, then it will be attached to a point in a tree T_k. The union of these paths forms a tree R_{ij} with root vertex z_{j0} , as in Fig. 19.

Then, we let X_i denote the union of the trees R_{ij} and the Y–network $\mathcal{Y}_i$, after attaching the end points of R_{ij} by the attaching maps. Then, the union of the R_{ij} and the T_k (after attaching the end points of R_{ij}) gives a maximal tree structure Q_i in X_i. Each curve segment (i.e. edge) in X_i and not in Q_i corresponds to a

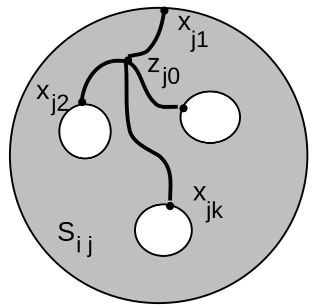

FIGURE 19. Tree structure R_{ij} in medial sheet S_{ij}

generator of $\pi_1(X_i)$. We choose one such z_{j0} and denote it by z_0. We choose paths α_j in Q_i from z_0 to z_{j0}. As Q_i is a tree the homotopy class of the α_j only depends on the end points.

We choose for each S_{ij} *without a medial edge curve*, an element of $\pi_1(S_{ij}, z_{j0})$ as follows. Construct for each boundary circle $C_{j\ell}$ which is attached to the Y–network a loop by following the path from z_{j0} to $x_{j\ell}$, then around the boundary circle $C_{j\ell}$ in counterclockwise direction, and then backwards along the path from $x_{j\ell}$ back to z_{j0}. We denote those generators by $\delta_{j\ell}$. We form $\delta_j = \delta_{j2} * \delta_{j3} * \cdots \delta_{jk} * \delta_{j1}^{-1}$ Then, the fundamental group condition takes the form.

3.6 (Fundamental Group Condition)**.** *For a contractible irreducible medial component M_i, the set of elements*

$$\{\alpha_j \cdot \delta_j \cdot \alpha_j^{-1} : \text{ all } j \text{ for which } S_{ij} \text{ is without an edge curve of } M_i\}$$

form a set of generators for $\pi_1(X_i, z_0)$.

4. Statement of the Structure Theorem for Contractible Regions

For contractible regions Ω, the structure theorem for the Blum medial axis takes the following form.

Theorem 4.1 (Structure Theorem for Contractible Regions)**.** *Suppose $\Omega \subset \mathbb{R}^3$ is a contractible bounded region with generic smooth boundary $\mathcal{B}$ and Blum medial axis M. Then, there is associated to Ω a multilevel directed tree structure which determines the structure of M up to homotopy equivalence in terms of attaching of irreducible medial components along fin curves.*

1. *At the top level $\Gamma(M)$ is a directed tree consisting of vertices corresponding to the irreducible medial components M_i of M; and there are directed edges from M_i to M_j corresponding to the attaching of an edge of M_i to M_j along a fin curve.*
2. *At the second level, to each M_i is associated a directed tree $\Lambda(M_i)$ with two types of vertices: S–vertices $\blacksquare$ corresponding to smooth medial sheets S_{ij} of M_i, and Y–nodes $\bullet$ corresponding to the connected components $\mathcal{Y}_{ik}$ of the Y–network of M_i. There is an edge from S_{ij} to $\mathcal{Y}_{ik}$ if a boundary "circle"of S_{ij} is attached to $\mathcal{Y}_{ik}$.*
3. *Each medial sheet S_{ij} is topologically a 2-disk with a finite number of holes. At most one of the boundary circles of S_{ij} is an edge curve of M_i. Each*

Y–network component $\mathcal{Y}_{ik}$ can be described as a 4-valent extended graph Π_{ik} (or by the corresponding reduced graph $\tilde{\Pi}_{ik}$).

(4) *At the third level, the tree $\Lambda(M_i)$ has data assigned to the S–vertices, Y–nodes, and edges. To an S–vertex for S_{ij} is assigned (h, e) indicating the number of holes and medial edge curves; to a Y–node for $\mathcal{Y}_{ik}$ is assigned the 4-valent extended graph Π_{ik}; and to an edge from S_{ij} to $\mathcal{Y}_{ik}$, the topological attaching data of the boundary circle of S_{ij} to $\mathcal{Y}_{ik}$.*

(5) *Furthermore, for an irreducible medial component M_i, the number of medial sheets, edge curves, 6-junction points and the total number of connected components of the Y–network $\mathcal{Y}_i$ satisfy the numerical relation (3.1), and M satisfies the fundamental group relation (3.6).*

Conversely, suppose we are given a bounded region Ω in $\mathbb{R}^3$ with smooth generic boundary and Blum medial axis M so that: the top level graph $\Gamma(M)$ is a tree; for each irreducible medial component M_i, the graph $\Lambda(M_i)$ is a tree; the medial sheets are topologically 2–disks with a finite number of holes, having at most one boundary circle an edge curve of M_i; the numerical invariants of M_i satisfy (3.1); and each M_i satisfies the fundamental group condition (3.6). Then, Ω is contractible region.

Example 4.2 (Simplest Examples)**.** As the simplest examples illustrating the structure theorem, we consider those for which the second level trees are $\Pi_i = \emptyset$ or $\circ$.

1) *"Simple Examples":* Each $\Pi_i = \emptyset$ so each M_i is topologically a 2–disk. This is the simplest type of region. For example, in Fig. 1, the medial axis consists of four irreducible medial components, each of which is of this type. The M-rep structures of Pizer and coworkers [P1] are based on the region having such a simple medial structure. In fact, they concentrate on regions which can be represented by a main medial component to which are attached other medial components so their top level trees have the form of a central vertex to which there are edges from all other vertices.

2) For the next simplest types of regions we allow both $\Pi_i = \emptyset$ or $\circ$ and the only type of medial sheets must be 2–disks and annuli. Besides the example in Fig. 1, we also see the example in Figure 6 and the first three examples of Fig. 16 are of this type.

3) The next case would have a reduced network graph of the form "$\bullet 4$"as for the fourth example of Fig. 16. In this example there is a single medial sheet which is an annulus.

Example 4.3 (Tree Structure for a Noncontractible Region)**.** It is possible for a Region to have tree structures at each of the two levels and yet not be contractible. An example is given by Fig. 20. The medial axis is irreducible, while the single second level graph Λ is a tree. Also, the medial sheets satisfy condition (3) of Theorem 4.1. However, M is not contractible. We can see this from the failure of the Euler relation: $s = 5$, $e = 2$, while $c = 2$ and $v = 0$, so $s - e = 3 \neq 2 = v + c$. In fact, M encloses a cavity in this case.

5. The General Classification

Suppose we now pass from a simpler contractible regions to more general regions Ω which we still suppose are bounded connected and have generic smooth boundaries $\mathcal{B}$. Now the local models for the singular points of the Blum medial axis M

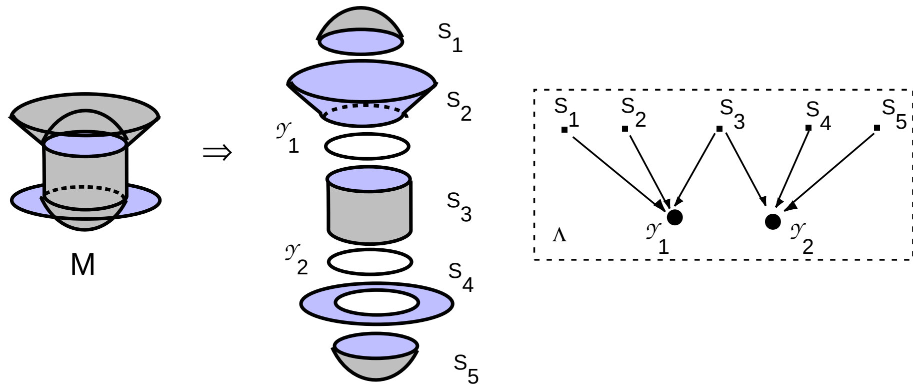

Figure 20. Medial Axis M of a Noncontractible Region with Tree Structures for $\Gamma(M)$ and $\Lambda(M_i)$

are still valid. Also, we can still decompose M into irreducible medial components as earlier. However, the resulting structures are no longer so simple.

(1) At the top level, $\Gamma(M)$ is a directed graph, but no longer need be a tree.
(2) At the second level the directed graph $\Lambda(M_i)$ still has the same general form of S–vertices and Y–nodes, with edges only going from S–vertices to$Y$–nodes. However, again the graph need not be a tree.
(3) Now each medial sheet S_{ij} is a compact surface with boundary, and can have varying genus, even be nonorientable, and have a multiple number of boundary circles which are edge curves of the medial axis.
(4) The components of the Y-network will still have the form of a 4-valent extended graph. However, there may be more than one boundary circle of S_{ij} being attached to the same component Y_{ik}.
(5) There are numerical relations between the the number of sheets, edge curves, etc; except now these relations also involve topological invariants of the medial components.
(6) Likewise, there is an analogue of the fundamental group relation, except now it involves the fundamental group of the medial component.

6. Summary and Conclusions

We introduce a method for describing the global structure of the Blum medial axis of a contractible region with smooth generic boundary in $\mathbb{R}^3$ in terms of a decomposition into "irreducible medial components"which are attached to each other along fin curves. The complete structure cannot be described by a simple tree structure as happens in the $2D$ case. Instead it involves an inductive process which specifies the attaching along fin curves. There are also several sources of nonuniqueness which we explain; and we indicate how external criteria can yield a unique structure for certain classes of objects.

There is a simplified medial structure topologically equivalent in a weak sense to the original that can be described by a two level tree structure. The top level tree structure determines how the irreducible medial components are attached. A second tree structure is associated to each medial component. This tree structure describes the structure of the medial component in terms of medial sheets which can be represented as 2–disks with a finite number of holes being attached to components of the Y–network. There is further data for each sheet and Y-network component. Furthermore, there are both numerical relations and fundamental group relations which must be satisfied. Conversely, these conditions together ensure that the original region is contractible.

References

[Bi] P. Bille *A survey on tree edit distance and related problems*, Theor. Comput. Sci. **337 no. 1-3** (2005), 217–239.

[Bg] I. Bogaevski *Metamorphoses of singularities of minimum functions and bifucations of shock waves of the Burgers equation with vanishing viscosity*, St. Petersburg (Leningrad) Math. Jour. **1 no. 4** (1990), 807–823.

[Bg2] ——— *Perestroikas of shocks and singularities of minimum functions*, Physica D **173** (2002), 1–28.

[BN] H. Blum and R. Nagel *Shape description using weighted symmetric axis features*, Pattern Recognition **10** (1978) 167–180.

[D1] J. Damon. *Smoothness and Geometry of Boundaries Associated to Skeletal Structures I: Sufficient Conditions for Smoothness*, Annales Inst. Fourier **53 no.6** (2003) 1945-1981

[D2] ——— *Determining the Geometry of Boundaries of Objects from Medial Data*, Int. Jour. Comp. Vision **63 (1)** (2005) 45–64

[D3] ——— *The Global Medial Structure of Regions in* $\mathbb{R}^3$, to appear in Geometry and Topology

[D4] ——— *Global Geometry of Regions and Boundaries via Skeletal and Medial Integrals*, to appear in Comm. Anal. and Geom.

[Ga] M. Gage *Curve shortening makes convex curves circular*, Invent. Math. **76** (1984), 357-364

[GH] M. Gage and R. Hamilton *The heat equation shrinking convex plane curves*, Jour. Diff. Geom. **23** (1986), 69–96

[Gb] P.J. Giblin, *Symmetry Sets and Medial Axes in Two and Three Dimensions* , The Mathematics of Surfaces, Roberto Cipolla and Ralph Martin (eds.), Springer-Verlag (2000), 306–321.

[GK] P.J. Giblin and B. Kimia *Transitions of the 3D Medial Axis Under a One-Parameter Family of Deformations*, Proc. ECCV 2002, Lecture Notes in Computer Science **2351** (2002), 718–734.

[Go] S. E. Goodman *Beginning Topology*, Brooks Cole Publ., Belmont CA (2005)

[Gr] M. Grayson *The heat equation shrinks embedded curves to round points*, Jour. Diff. Geom. **26** (1987), 285–314

[KTZ] B. B. Kimia, A. Tannenbaum, and S. Zucker *Toward a computational theory of shape: An overview*, O. Faugeras ed., Three Dimensional Computer Vision , M I T Press, 1990.

[Ma] J. Mather *Distance from a manifold in Euclidean space*, in Proc. Symp. Pure Math. vol **40** Pt 2 (1983), 199–216

[Mu] J. Munkres *Topology*, second edition, Prentice Hall Publ., Englewood Cliffs, NJ (2000)

[P1] S. Pizer et al *Deformable M-reps for 3D Medical Image Segmentation*, Int. Jour. Comp. Vision **55 no. 2-3** (2003), 85–106

[P2] S. Pizer et al *Multiscale Medial Loci and their Properties*, Int. Jour. Comp. Vision **55 no. 2-3** (2003), 155–179

[PS] S. Pizer and K. Siddiqi, editors, *Medial Representations: Mathematics, Algorithms, and Applications*, to appear Springer-Verlag Series on Computational Imaging and Vision

[SB] K. Siddiqi, S. Bouix, A. Tannenbaum and S. Zucker *The Hamilton–Jacobi Skeleton*, Int. Jour. Comp. Vision **48** (2002), 215–231

[Sn] J. Sethian *Level Set Methods* Cambridge Univ. Press (1996)

[SN] G. Szekely, M. Naf, Ch. Brechbuhler, and O. Kubler *Calculating 3d Voronoi diagrams of large unrestricted point sets for skeleton generation of complex 3d shapes*, Proc. 2nd Int. Workshop on Visual Form, World Scientific Publ. (1994), 532–541

[V] G. Valiente *Algorithms on Trees and Graphs* Springer-Verlag, Berlin (2002)

[Y] J. Yomdin *On the local structure of the generic central set*, Compositio. Math. **43** (1981), 225–238

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