Swept Regions and Surfaces: Modeling and Volumetric Properties

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To Andre Galligo who is both an excellent mathematician and an even better friend

Abstract

We consider "swept regions" Ω and "swept hypersurfaces" $\mathcal{B}$ in $\mathbb{R}^{n+1}$ (and especially $\mathbb{R}^3$) which are s disjoint union of subspaces $\Omega_t = \Omega \cap \Pi_t$ or $\mathcal{B}_t = \mathcal{B} \cap \Pi_t$ obtained from a varying family of affine subspaces $\{\Pi_t : t \in \Gamma\}$. We concentrate on the case where Ω and $\mathcal{B}$ are obtained from a skeletal structure (M, U). This generalizes the Blum medial axis M of a region Ω, which consists of the centers of interior spheres tangent to the boundary $\mathcal{B}$ at two or more points, with U denoting the vectors from the centers of the spheres to the points of tangency. We extend methods developed for skeletal structures so they can be deduced from the properties of the individual intersections Ω_t or $\mathcal{B}_t$ and a relative shape operator S_{rel}, which we introduce to capture changes relative to the varing family $\{\Pi_t\}$.

We use these results to deduce modeling properties of the global $\mathcal{B}$ in terms of the individual $\mathcal{B}_t$, and determine volumetric properties of regions Ω expressed as global integrals of functions g on Ω in terms of iterated integrals over the skeletal structure of Ω_t then integrated over the parameter space Γ.

Key words: swept regions, swept surfaces, Blum medial axis, swept skeletal structures, Whitney stratified sets, radial shape operator, relative shape operator, skeletal integrals

Introduction

Let $\Omega \subset \mathbb{R}^{n+1}$ be a region with boundary $\mathcal{B}$, or let $\mathcal{B}$ denote a hypersurface. Considerable recent work has made use of medial representations for Ω and $\mathcal{B}$ for solving a variety of computer imaging problems, see e.g. the survey [P] and the book [PS]. Skeletal structures provide a generalized form of medial structure, which includes both the Blum medial axis [BN] and generalized offset (hyper)surfaces, and can be used to analyze the chordal locus models of Brady and Asada, [BA] and arc-segment medial axis of Leyton [Le].

The Blum medial axis M of a region Ω with smooth generic boundary $\mathcal{B}$, consists of the locus of centers of spheres contained in Ω and tangent at two or more points (or with degenerate tangency). On M is a multivalued vector field U from points on M to the points of tangency. If we appropriately relax the conditions required for (M, U), we still obtain a "skeletal structure" (see [D1]). These skeletal struc-

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tures have been used to analyze the smoothness of $\mathcal{B}$ and determine the local, relative, and global geometry of Ω and $\mathcal{B}$ using "radial and edge shape operators" defined for these skeletal structures (see [D1] - [D6], and Chap 3 of [PS]).

In this paper we consider skeletal structures for regions or hypersurfaces which are "swept-out" by a family of subspaces (see e.g. figure 1). For such regions, we shall see how we may exploit the swept structure to compute the corresponding mathematical operators and apply the preceding results to determine smoothness and geometric properties of such regions or hypersurfaces. Although the immediate applications for imaging are for regions and their boundaries in $\mathbb{R}^3$, we carry out the computations for arbitrary dimensions, demonstrating the general form of these results.

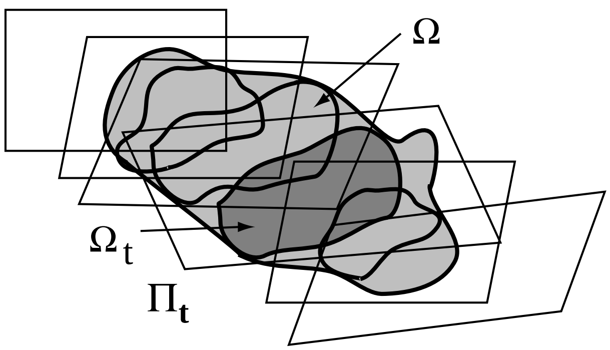

Fig. 1. Swept region Ω by a family of varying affine subspaces Π_t and associated swept boundary

Specifically, we consider a *swept decomposition* of the region Ω or the hypersurface $\mathcal{B}$ which is obtained by the intersection of Ω or $\mathcal{B}$ with a family of $(n-k+1)$–dimensional affine subspaces Π_t (parametrized by a k–dimensional submanifold Γ) so that Ω, resp. $\mathcal{B}$, is a disjoint union $\Omega_t = \Omega \cap \Pi_t$, resp. $\mathcal{B}_t = \mathcal{B} \cap \Pi_t$. Then, we refer to Ω as a *swept region* or $\mathcal{B}$ as a *swept (hyper)surface*.

Conversely, for modeling purposes we may ask when a family of $(n-k)$–dimensional smooth manifolds $\mathcal{B}_t \subset \Pi_t$, which are defined using skeletal structures (M_t, U_t) in Π_t, together will form a smooth hypersurface $\mathcal{B}$. The answer depends on both geometric properties of the $\mathcal{B}_t$ and the variation properties of the family of affine subspaces $\{\Pi_t\}$.

A second question concerns "volumetric properties" of such a swept region Ω. Such properties are given by various global geometric invariants of Ω which can be expressed as integrals over regions of Ω. We will use the computations of the operators together with integral results from [D4] to give general expressions for these integrals as iterated integrals over either the family of skeletal structures (M_t, U_t) or over the swept decomposition $\{\Omega_t\}$ of Ω. For example, such integral representations have been used to provide criteria for matching objects in a population, see [TG], [T].

A third question concerns regions such as "irregular tube-like" structures. A tube-like structure should be representable by a series of slices through it; however, the irregularity means that there is no natural center curve for the tube so the slices are unlikely to be orthogonal to any chosen central curve (or alternative medial structure). A medial-type representation using a central curve leads to a notion of a "contracted medial structure", which involves a lower dimensional skeletal set, with a complementary dimensional family of radial vectors. We also answer the corresponding questions concerning smoothness and volumetric properties for (hyper)surfaces and regions defined by these structures.

To answer these questions, we introduce a synthesis of these two ideas of swept regions and surfaces and the skeletal representations, to deduce modeling properties of swept surfaces and deduce formulas for global integral properties of such swept regions. Specifically, for a swept surface $\mathcal{B}$ (and swept region Ω) we consider the case when we have a skeletal representation of each Ω_t and $\mathcal{B}_t$ by (M_t, U_t) so that $M = \cup_t M_t$ and $U = \cup_t U_t$ defines a skeletal structure for Ω and $\mathcal{B}$. We shall refer to this as a *swept skeletal structure*. Note that even if each M_t is the Blum medial axis of Ω_t, then (M, U) will in general only be a skeletal structure. To capture the geometric properties in such situations, we shall introduce a "relative shape operator" which measures how U varies in the complementary direction to Π_t.

First, for a swept skeletal structure (M_t, U_t), we will determine the associated radial shape operator associated to (M, U) in terms of the radial shape operators $S_{rad}(t)$ for each (M_t, U_t) together with the "relative shape operator" S_{rel}. Second, in the case of swept surfaces in $\mathbb{R}^3$, we show in Proposition 2.8 that the principal edge curvature κ_E (the generalized eigenvalue of the edge shape operator) equals the relative principal curvature κ_{rel} (which gives the relative shape operator in this case). Third, using the preceding and the results from [D1] and [D3], we deduce sufficient conditions (Theorem 3.1) for the smoothness of the associated boundary surface $\mathcal{B}$ in $\mathbb{R}^3$ (given the smoothness of each $\mathcal{B}_t$) solely in terms of κ_{rel}. This has been applied to modeling crest regions of surfaces [HPD] and smoothness of models for more general skeletal structures in [H]. Fourth, we also apply these results to a contracted skeletal structure allowing, e.g. a formulation for properties of irregular- type generalized tubes along a skeletal curve. Lastly, we also apply the results from [D4] to express integrals of functions over (subregions of) Ω as iterated integrals of functions over regions in each Π_t, then integrated over Γ with respect to a kernel computed from the relative shape operator (Theorems 4.5 and 4.12).

CONTENTS

1. Swept Representations of Regions, Hypersurfaces, and Skeletal Structures

Swept Regions and Swept Hypersurfaces

Suppose that $\Omega \subset \mathbb{R}^{n+1}$ is a compact region with smooth generic boundary $\mathcal{B}$, or more generally $\mathcal{B}$ is a hypersurface in $\mathbb{R}^{n+1}$. We define what we mean by smooth families of affine subspaces, and then by Ω or $\mathcal{B}$ being represented by such a smooth family as a swept region or swept hypersurface.

Definition 1.1 *A parametrized family* $\{\Pi_t\}_{t\in\Gamma}$, *for* Γ *a* k*–dimensional submanifold of* $\mathbb{R}^{n+1}$*, will be called a* smooth family of $(n-k+1)$–dimensional affine subspaces *of* $\mathbb{R}^{n+1}$ *if there is an* $(n-k+1)$*–dimensional vector bundle* E *on* Γ *and a smooth map* $\gamma : E \to \mathbb{R}^{n+1}$ *such that on each fiber* $\gamma_t : E_t \to \mathbb{R}^{n+1}$ *is an affine embedding with* $\Pi_t = \gamma_t(E_t)$ *transverse to* Γ *at* $t \in \Gamma$ *.*

Definition 1.2 *A region* Ω *is represented as a* swept region *by the smooth family of* $(n-k+1)$*–dimensional affine subspaces* $\{\Pi_t\}_{t\in\Gamma}$ *if:*
i) any point in Ω *lies in exactly one* Π_t*;*
ii) the map $\gamma : E \to \mathbb{R}^{n+1}$ *defining the family* Π_t *is a diffeomorphism on* $\gamma^{-1}(\Omega)$*; and*
iii) if we identify Γ *with the zero section of* E*, then the map* $\gamma : \Gamma \to \mathbb{R}^{n+1}$ *is transverse to each* Π_t *at all points of* Ω*.*
We say $\{\Pi_t\}_{t\in\Gamma}$ *is a smooth swept family on* Ω*.*

Likewise, a hypersurface $\mathcal{B}$ *is represented as a* swept hypersurface *by* $\{\Pi_t\}_{t\in\Gamma}$ *if in the preceding, the conditions hold with* Ω *replaced by* $\mathcal{B}$*.*

We shall frequently identify Γ with its image in $\mathbb{R}^{n+1}$.

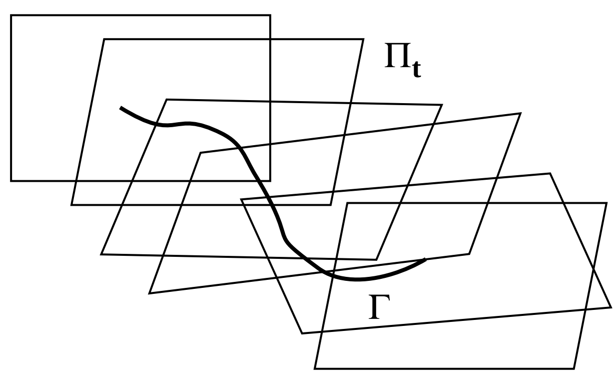

Fig. 2. Smooth Family of affine spaces along the manifold Γ

If a region Ω with smooth boundary is represented as a swept region, then its boundary is represented as a swept surface. If a hypersurface $\mathcal{B}$ is compact, then by the parametrized transversality theorem and the openness of transversality, the set of $t \in \Gamma$ such that Π_t is transverse to $\mathcal{B}$ is open and dense in Γ, with complement of measure zero. For such t, $\mathcal{B}_t = \mathcal{B} \cap \Pi_t$ is a smooth manifold of dimension $n - k$ and there is an open dense subset of $\mathcal{B}$ which belongs to the union of such manifolds. If $\mathcal{B}$ is the boundary of the compact region Ω which is represented as a swept region by $\{\Pi_t\}_{t \in \Gamma}$, then for an open dense set of $t \in \Gamma$, $\Omega_t = \Omega \cap \Pi_t$ is a region in Π_t with smooth boundary $\mathcal{B}_t$. Then, there is open dense subset of Ω, which is a union of such regions, whose complement in Ω has measure zero.

Remark 1.3 *In all that follows, we will on numerous occasions also consider local versions of swept representations of objects, which hold on an open set. We will frequently refer to such* local swept representations *without further discussion.*

Example 1.4 *The simplest example is when all Π_t are parallel translates and Γ is a linear subspace orthogonal to the Π_t. Then, we are just taking parallel slices.*

A second example is when Γ is a smooth $k-$dimensional submanifold of $\mathbb{R}^{n+1}$ and Π_t is the orthogonal affine complement to $T_t\Gamma$ and passing through t. At least in a tubular neighborhood of Γ we know $\{\Pi_t\}_{t \in \Gamma}$ is a smooth swept family. More generally we can replace the orthogonal complement by a smoothly varying family of complementary subspaces. In §3, we consider the situation of a curve $\Gamma \subset \mathbb{R}^3$, with a smoothly varying family of complementary planes.

Skeletal Structures

We next recall the notion of a "skeletal structure" (M, U) in $\mathbb{R}^{n+1}$ introduced in [D1] (or see the less technical discussion in [D3]). It consists of the *skeletal set* M which is a Whitney stratified set satisfying certain special conditions. On M is the *radial vector field* U which is a multivalued vector field where the number of values vary depending on the stratum. Furthermore, M and U satisfy certain extra conditions which always are satisfied for Blum medial axes (see [D1, §1] for a complete discussion).

A Whitney stratified set M may be represented as a union of disjoint smooth strata M_α of varying dimensions satisying the "axiom of the frontier"(if $M_\beta \cap \bar{M}_\alpha \neq \emptyset$, then $M_\beta \subset \bar{M}_\alpha$); and Whitney's conditions a) and b) (which involve limiting properties of tangent planes and secant lines). Key properties of Whitney statified sets are found in [M1] and [Gi], and are summarized in [D1, §1]. For example, for regions Ω with smooth generic boundaries $\mathcal{B}$, the Blum medial axis is a Whitney stratified set by Mather [M2]). Its local structure has been determined by Yomdin [Y], Mather [M2], and Giblin [Gb] for an explicit geometric description for regions in $\mathbb{R}^3$, and it satisfies the other extra conditions, see e.g. [BN], [P], and Chap. 2 by Giblin-Kimia in [PS].

We let M_{reg} denote the points in the top dimensional strata (this is the dimension n of M and these points are the "smooth points"of M). Also, we let M_{sing} denote the union of the remaining strata, and ∂M denote the subset of M_{sing} consisting of the "edge points"of M at which M is locally an n manifold with boundary, with the points being boundary points. An important property of a skeletal structure is that each local component of M_{reg} has a unique limiting tangent space as we approach any point in M_{sing} from that component.

For example, for regions in $\mathbb{R}^3$ with smooth generic boundary, the types of points of M are shown in figure 3.

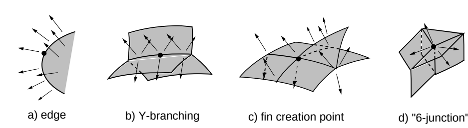

Fig. 3. Local generic structure for Blum Medial axes in $\mathbb{R}^3$ and the associated Radial Vector Fields

Swept Skeletal Structures

Suppose that (M, U) is a skeletal structure with associated boundary $\mathcal{B}$ which encloses the region Ω. Suppose also that Ω is represented as a swept region via the smooth family $\{\Pi_t\}_{t \in \Gamma}$ of $(n - k + 1)$–dimensional affine subspaces.

Definition 1.5 *We say that (M, U) is a* swept skeletal structure *if for each $x \in M$ with say $x \in \Pi_t$, and for each value $U(x)$ of U at x, $U(x) \in \Pi_t$. We then refer to the resulting associated boundary $\mathcal{B}$ as a radial swept hypersurface.*

Again by the parametrized transversality theorem and the openness of transversality to closed Whitney stratified sets, the set of $t \in \Gamma$ such that Π_t is transverse to M (i.e. the strata of M) is open and dense in Γ, with complement of measure zero. For such t, $M_t = M \cap \Pi_t$ is a Whitney stratified set of dimension $n-k$. In addition, if U_t denotes the restriction of U to M_t, then for an open dense subset of $t \in \Gamma$, (M_t, U_t) is a skeletal structure with associated boundary $\mathcal{B}_t$ enclosing the region $\Omega_t = \Omega \cap \Pi_t$ in Π_t.

Contracted Skeletal Structures

A skeletal structure (M, U) in $\mathbb{R}^{n+1}$ will have the skeletal set of dimension n. There are situations where due to symmetry (such as cylindrical symmetry in $\mathbb{R}^3$), the Blum medial axis will be a curve. For regions in $\mathbb{R}^3$ such as irregular tubes which are close to having such a symmetry, there may be advantages to representing them medially using a curve. We allow such a situation by introducing a *contracted skeletal structure*.

Definition 1.6 *A* contracted skeletal structure *will consist of a compact k–dimensional Whitney stratified set $M \subset \mathbb{R}^{n+1}$, an n–dimensional Whitney stratified set $\tilde{M}$ (in some other manifold), a stratified map $p : \tilde{M} \to M$, and a vector field $U : \tilde{M} \to \mathbb{R}^{n+1}$ along the map p so that there is an $\varepsilon > 0$*

(i) the map $\Psi : \tilde{M} \times \mathbb{R} \to \mathbb{R}^{n+1}$ defined by $\Psi(x, t) = p(x) + t \cdot U(x)$, when restricted to $\tilde{M} \times (0, \varepsilon]$, is a homeomorphism onto its image (and is a diffeomorphism restricted to the closure of each stratum of $\tilde{M}$);

(ii) $U(\Psi(x, t))$ is transverse to $\Psi(\tilde{M} \times \{t\})$ at smooth points and to all of the limiting tangent planes at points coming from the singular points of M; and

(iii) $\Psi(\tilde{M} \times (0, \varepsilon)) \cup M$ is a neighborhood of M.

For such contracted skeletal structures, $\Psi(\tilde{M} \times (0, \varepsilon)) \cup M$ is a "tubular neighborhood"of M. By [D1, Thm 5.1], any skeletal structure (M, U) satisfies definition 1.6. We can view the map Ψ as a "radial flow"from M filing out the tubular neighborhood, which is fibered by the level sets $\psi_t(\tilde{M})$, where $\psi_t(x) = \Psi(x, t)$.

Just as for a skeletal structures, for a contracted skeletal structure $(M, \tilde{M}, U)$, we can introduce the "region"$\Omega = \Psi(\tilde{M} \times (0, 1]) \cup M$, and its "associated boundary"$\mathcal{B} = \psi_1(\tilde{M})$. In general, Ω need not be a region, nor will $\mathcal{B}$ be its piecewise smooth boundary. In the case of skeletal structures, Theorem 2.5 of [D1] gives a criterion for the smoothness of $\mathcal{B}$ as a smooth boundary of the region Ω. We provide a criteria for this more general case in §3.

Example 1.7 (Polar Swept Hypersurfaces) *The simplest example of a contracted skeletal structure consists of a k–dimensional submanifold M of $\mathbb{R}^{n+1}$ with a smooth family of swept complementary $(n+1-k)$–dimensional affine subspaces $\{\Pi_t\}_{t \in M}$ defined via a vector bundle E, with $\tilde{M}$ = unit sphere bundle in E, and $U = r \cdot U_1$, where U_1 is the unit radial vector field in Π_t, and r is a positive function on the unit sphere bundle in E.*

For $x \in M$, and $\tilde{M}_{(x)}$ denoting the unit sphere in $\tilde{M}$ over x, the radial vector field at all points of $\Psi(\tilde{M}_{(x)})$ lies in the image of $E_{(x)}$, namely, $\{\Pi_t\}$. Then, $\mathcal{B}_x = \Psi(\tilde{M}_{(x)})$, is given by the radial function r for polar coordinates for the unit sphere $\tilde{M}_{(x)} \subset \Pi_t$.

We refer to the resulting associated boundary $\mathcal{B}$ as a polar swept hypersurface. When M is the image of a curve $\gamma(t)$ in $\mathbb{R}^3$, we obtain a generalized tube about the curve $\gamma(t)$, where the slices are affine rather than normal slices, and the curve in each slice ($\mathcal{B}_{\gamma(t)}$) varies as t varies (see figure 4).

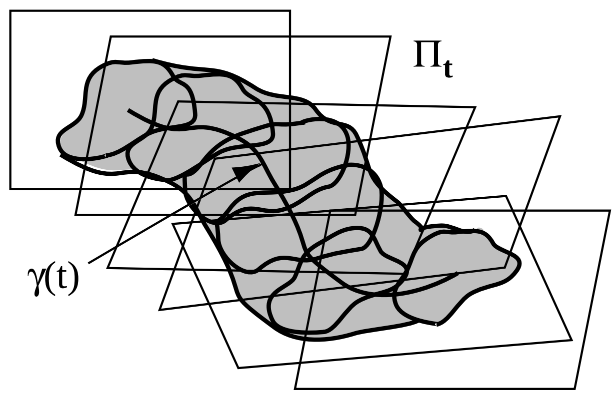

Fig. 4. Polar swept (hyper)surface swept by a smooth family of planes Π_t along the curve $\gamma(t)$.

2. Relative Shape Operators

Before defining relative shape operators for swept skeletal structures, we first recall the definition of radial shape operators associated to skeletal structures.

Radial Shape Operators and Principal Radial Curvatures

Given a skeletal structure (M, U) in $\mathbb{R}^{n+1}$, we consider for a regular point x_0 a choice of a smooth value of U defined in a neighborhood of x_0. We may represent $U = r \cdot U_1$, for an associated unit vector field U_1. Then, the *radial shape operator* is defined by

$$S_{rad}(v) \quad = \quad -\mathrm{proj}_U(\frac{\partial U_1}{\partial v})$$

for $v \in T_{x_0}M$. Here proj_U denotes projection onto $T_{x_0}M$ along U (which in general is not orthogonal to $T_{x_0}M$). Then, $S_{rad} : T_{x_0}M \to T_{x_0}M$ is linear but not necessarily symmetric. We call the eigenvalues of S_{rad} the *principal radial curvatures* at x_0, and denote them by $\kappa_{r\,i}$.

Given a basis $\{v_1, \ldots, v_n\}$ for $T_{x_0}M$, then for each i we may represent

$$\frac{\partial U_1}{\partial v_i} \quad = \quad a_i \cdot U_1 - \sum_{j=1}^{n} s_{ji} v_j. \tag{1}$$

This equation can be written in vector form. We let $\mathbf{v}$ denote the column vector with i–th entry v_i, $A_{\mathbf{v}}$ with i–th entry a_i, $\frac{\partial U_1}{\partial \mathbf{v}}$ with i–th entry $\frac{\partial U_1}{\partial v_i}$. Also, $S_{\mathbf{v}}$ is the matrix with ij–th entry s_{ij} and is a matrix representation for S_{rad} with respect to the basis $\{v_1, \ldots, v_n\}$. Then, (1) can be written in vector form by

$$\frac{\partial U_1}{\partial \mathbf{v}} \quad = \quad A_{\mathbf{v}} \cdot U_1 - S_{\mathbf{v}}^T \cdot \mathbf{v} \tag{2}$$

In this equation we interpret $A_{\mathbf{v}} \cdot U_1$ as the column vector with i–th entry the vector $a_i \cdot U_1$; while $S_{\mathbf{v}}^T \cdot \mathbf{v}$ denotes the column matrix obtained by matrix multiplication of the scalars in $S_{\mathbf{v}}^T$ (the transpose of $S_{\mathbf{v}}$) times the vectors in $\mathbf{v}$.

Remark 2.1 *We emphasize that because there are two smooth values of U at smooth points, we obtain two shape operators at each point. Moreover, near a non–edge point $x_0 \in M_{sing}$, for each local smooth component of M_{reg} for x_0, each smooth value of U will extend smoothly to x_0. Thus, to each value of U and each local component, such a shape operator will be defined at x_0. Hence, any statement involving the shape operator will involve all of these for each point.*

Relative Shape Operators

Now we consider the case of a swept skeletal structure structure (M, U) by a smooth family of $(n+1-k)$–dimensional affine subspaces $\{\Pi_t\}_{t\in\Gamma}$ with the Π_t transverse to M in a neighborhood of a smooth point $x_0 \in M$. Then, if $x_0 \in \Pi_{t_0}$, for t in a neighborhood of t_0, $M_t = M \cap \Pi_t$, (M_t, U_t) defines a skeletal structure in Π_t smooth in a neighborhood of x_0. Hence, for each smooth value of U_t locally near x_0, there is defined a radial shape operator $S_{rad}(M_t)$.

We now proceed to define a *relative shape operator* for the entire skeletal structure (M, U). We again write $U = r \cdot U_1$ with U_1 a unit vector field. The relative shape operator will now measure how U_1 changes relative to the family of affine subspaces Π_t as we move along M in a direction tranverse to M_{t_0}. As M_{t_0} is smooth near x_0 in M, we may choose a complementary subspace N_{x_0} to $T_{x_0}M_{t_0}$ in $T_{x_0}M$. As Π_{t_0} is transverse to M at x_0, N_{x_0} is also a complementary subspace to Π_{t_0} in $\mathbb{R}^{n+1}$. As M_{t_0} has codimension k in M, N_{x_0} has dimension k. Then, for the smooth value of U, we define the *relative shape operator*

$$S_{rel} : N_{x_0} \to N_{x_0}$$

as follows:

$$S_{rel}(v) \quad = \quad -\mathrm{proj}_{\Pi_{t_0}}(\frac{\partial U_1}{\partial v})$$

where $\mathrm{proj}_{\Pi_{t_0}}$ denotes the projection onto N_{x_0} along Π_{t_0} (recall $x_0 \in \Pi_{t_0}$).

First, we claim

Lemma 2.2 *Up to conjugacy, S_{rel} is independent of the choice of N_{x_0}.*

Proof: Let N'_{x_0} be another complementary subspace to $T_{x_0}M_{t_0}$ in $T_{x_0}M$. Also, we let α denote the restriction to N'_{x_0} of the projection from $\mathbb{R}^{n+1}$ to N_{x_0} along Π_{t_0}. Then, $\alpha : N'_{x_0} \simeq N_{x_0}$. Given $v' \in N'_{x_0}$, we let $v = \alpha(v')$. Thus, $v' - v = w \in T_{x_0}M_{t_0}$. Since $U_1 \in \Pi_{t_0}$ for all $x \in M_{t_0}$, if $w \in T_{x_0}M_{t_0}$ $\frac{\partial U_1}{\partial w} \in \Pi_{t_0}$. Hence,

$$\frac{\partial U_1}{\partial v'} \quad = \quad \frac{\partial U_1}{\partial v} \mod \Pi_{t_0}$$

Hence, applying minus the projection $\mathrm{proj}_{\Pi_{t_0}}$ onto N_{x_0} along Π_{t_0}, we obtain

$$-\mathrm{proj}_{\Pi_{t_0}}(\frac{\partial U_1}{\partial v'}) \quad = \quad -\mathrm{proj}_{\Pi_{t_0}}(\frac{\partial U_1}{\partial v}) \tag{3}$$

If instead $\mathrm{proj}'_{\Pi_{t_0}}$ denotes projection onto N'_{x_0} along Π_{t_0}, then

$$\alpha \circ \mathrm{proj}'_{\Pi_{t_0}} \quad = \quad \mathrm{proj}_{\Pi_{t_0}}$$

Hence (3) becomes

$$\alpha \circ \mathrm{proj}'_{\Pi_{t_0}}(\frac{\partial U_1}{\partial v'}) \quad = \quad \mathrm{proj}_{\Pi_{t_0}}(\frac{\partial U_1}{\partial v})$$

With S'_{rel} denoting the relative shape operator computed using N'_{x_0}, we obtain

$$\alpha \circ S'_{rel} \circ \alpha^{-1}(v) \quad = \quad S_{rel}(v)$$

for all $v \in N_{x_0}$, as claimed. $\square$

Hence, the eigenvalues of S_{rel} are well-defined. We denote them by $\kappa_{rel,j}$ and call them the *principal* *relative curvatures* of the swept skeletal structure representation.

Second, we may obtain a matrix representation for S_{rel} in an analogous fashion as for S_{rad}. We choose a basis $\{v_1, \ldots, v_k\}$ for N_{x_0} and for each i represent

$$\frac{\partial U_1}{\partial v_i} \quad = \quad w_i - \sum_{j=1}^{k} s_{ji} v_j \tag{4}$$

where $w_i \in \Pi_{t_0}$. This equation can be written in vector form analogous to (2). We let $\mathbf{v}$ denote the column vector with i–th entry v_i, $\mathbf{w}$ with i–th entry w_i, $\frac{\partial U_1}{\partial \mathbf{v}}$ with i–th entry $\frac{\partial U_1}{\partial v_i}$. Also, $S_{rel,\mathbf{v}}$ is the matrix with ij–th entry s_{ij} and is a matrix representation for S_{rel} with respect to the basis $\{v_1, \ldots, v_k\}$. Then, (1) can be written in vector form by

$$\frac{\partial U_1}{\partial \mathbf{v}} \quad = \quad \mathbf{w} - S_{rel,\mathbf{v}}^T \cdot \mathbf{v} \tag{5}$$

In this equation, $S_{rel,\mathbf{v}}^T \cdot \mathbf{v}$ denotes the column matrix obtained by matrix multiplication of the scalars in $S_{rel,\mathbf{v}}^T$ (the transpose of $S_{rel,\mathbf{v}}$) times the vectors in $\mathbf{v}$.

Example 2.3 (Swept Skeletal Structures in $\mathbb{R}^3$) *We next consider the special case of a swept skeletal structure (M, U) in $\mathbb{R}^3$, given by a smooth family of planes $\{\Pi_t\}_{t\in\Gamma}$ with Γ a curve. Then, N_{x_0} is a line, and for nonzero $v \in N_{x_0}$, (5) becomes*

$$\frac{\partial U_1}{\partial v} \quad = \quad w - \kappa_{rel} \cdot v \qquad \text{with } w \in \Pi_{t_0} \tag{6}$$

The relative shape operator is just multiplication by κ_{rel}, and κ_{rel} is the principal relative curvature.

Computing Relative Principal Curvatures without Normalizing U

It is possible to compute the radial shape and edge operators, without having to first normalize U to the unit vector field U_1 (see e.g. [PS, Chap. 3]). For example,

$$-\text{proj}_U\left(\frac{\partial U}{\partial v}\right) \quad = \quad r \cdot S_{rad}(v)$$

Thus, $r \cdot S_{rad}$ can be computed without normalizing. It has eigenvalues $\{r\kappa_{r,i}\}$, and the conditions such as smoothness of the boundary are expressed in terms of the $r\kappa_{r,i}$. In an exactly analogous fashion, we can compute the relative shape operator

$$-\text{proj}_{\Pi_{t_0}}\left(\frac{\partial U}{\partial v}\right) \quad = \quad r \cdot S_{rel}(v)$$

We shall see that conditions involving $r \cdot S_{rad}$ and its eigenvalues $\{r\kappa_{r,i}\}$ can then be expressed in terms of $r \cdot S_{rel}$ and its eigenvalues $\{r\kappa_{rel,i}\}$.

Remark 2.4 *If the singular point $x \in M_{sing}$ is not an edge point, then for each local smooth component M_i in a neighborhood of x, and smooth value of U on M_i, we can analogously define a relative shape operator at x.*

Radial Shape Operator from Relative Shape Operator

Next, we show how to determine for a swept skeletal structure (M, U), the matrix representation for the radial shape operator in terms of the radial shape operators for the slices and the relative shape operator.

In addition to the basis $\mathbf{v}$ for N_{x_0}, we also choose a basis $\mathbf{v}' = \{v'_1, \ldots, v'_{n-k}\}$ for $T_{x_0}M_{t_0}$. Together $\mathbf{v}'$ and $\mathbf{v}$ give us a basis $\mathbf{v}''$ for $T_{x_0}M$. Then, we may compute the matrix representation of S_{rad} for (M, U) in terms of $S_{rad,\mathbf{v}'}(M_{t_0})$ and $S_{rel,\mathbf{v}}$.

Proposition 2.5 *The matrix representation of S_{rad} with respect to the basis $\mathbf{v}''$ is given by*

$$S_{rad,\mathbf{v}''} \quad = \quad \begin{pmatrix} S_{rad,\mathbf{v}'}(M_{t_0}) & * \\ 0 & S_{rel,\mathbf{v}} \end{pmatrix} \tag{7}$$

Proof: Since $U_1 \in \Pi_{t_0}$ for all $x \in M_{t_0}$, if $w \in T_{x_0}M_{t_0}$, then $\frac{\partial U_1}{\partial w} \in \Pi_{t_0}$. Furthermore, if we apply $-\text{proj}_{U_{t_0}}$, then we obtain $S_{rad}(M_{t_0})(w)$. Hence the first $n-k$ columns of $S_{rad,\mathbf{v}''}$ have the desired form.

Second, if $w \in N_{x_0}$, then

$$-\text{proj}_U\left(\frac{\partial U_1}{\partial w}\right) \quad = \quad S_{rel}(w) + w' \tag{8}$$

where $w' \in T_{x_0}M_{t_0}$. Thus, writing the RHS of (8) in terms of the basis $\mathbf{v}''$ implies that the last k columns of $S_{rad,\mathbf{v}''}$ have the form given by the RHS of (7). □

We immediately deduce several corollaries from the block upper triangular form of $S_{rad,\mathbf{v}''}$ in (7).

Corollary 2.6 *For a swept skeletal structure, the principal radial curvatures for the smooth value U at x_0 consists of the union of the principal radial curvatures for (M_{t_0}, U_{t_0}) at x_0 and the principal relative curvatures at x_0, counting multiplicities:*

$$\{\kappa_{rad,i}\} \quad = \quad \{\kappa_{rad,j}(M_{t_0})\} \cup \{\kappa_{rel,\ell}\} \tag{9}$$

Second, we deduce the form of the determinants of S_{rad} and $I - tr \cdot S_{rad}$ (for skeletal integral formulas given in §4).

Corollary 2.7 *For a swept skeletal structure, there are the following formulas for determinants at $x_0 \in \Pi_{t_0}$:*

$$\det(S_{rad}) \quad = \quad \det(S_{rad}(M_{t_0})) \cdot \det(S_{rel}) \qquad (10)$$

and

$$\det(I - tr \cdot S_{rad}) \; = \; \det(I - tr \cdot S_{rad}(M_{t_0})) \cdot \det(I - tr \cdot S_{rel}) \qquad (11)$$

Relative Principal Curvature and Principal Edge Curvature

If (M, U) is a skeletal structure, then for points of ∂M, U is tangent to M, so the radial shape operator is not defined. In its place is the Edge– shape operator. Given a point $x_0 \in \partial M$ and a smooth value U at x_0, we let $\mathbf{n}$ be the unit normal vector field to M in a neighborhood of x_0. Then, we define the *Edge–shape operator* by

$$S_E(v) \quad = \quad -\text{proj}'(\frac{\partial U_1}{\partial v})$$

for $v \in T_{x_0}M$. Here proj$'$ denotes projection onto $T_{x_0}\partial M \oplus < \mathbf{n} >$ along U_1.

Given a basis $\{v_1, \dots, v_{n-1}\}$ of $T_{x_0}\partial M$, we also choose a vector v_n in the edge coordinate system at x_0 so that $\{v_1, \dots, v_{n-1}, v_n\}$ is a basis $T_{x_0}M$ in the edge coordinate system and so that v_n maps under the edge parametrization map to $c \cdot U_1(x_0)$ where $c \geq 0$ (the specific value of c is immaterial). Then, we can compute a matrix representation $S_{E\,\mathbf{v}}$ for S_E in a manner analogous to (2) using the bases $\{v_1, \dots, v_{n-1}, v_n\}$ in the domain and $\{v_1, \dots, v_{n-1}, \mathbf{n}\}$ in the range, where $\mathbf{n}$ is a unit normal vector field to M on a neighborhood W of x_0.

The *principal edge curvatures* of M at x_0 are the generalized eigenvalues of $(S_{E\,\mathbf{v}}, I_{n-1,1})$, where $I_{n-1,1}$ denotes the $n \times n$–diagonal matrix with 1's in the first $n-1$ diagonal positions and 0 otherwise. (recall the generalized eigenvalues of an ordered pair (A, B) of $n \times n$–matrices consists of λ such that $A - \lambda \cdot B$ is singular). The generalized eigenvalues of $(S_{E\,\mathbf{v}}, I_{n-1,1})$ are called the *principal edge curvatures of* M and we denote them by $\{\kappa_{E\,i}\}$ (note the number of generalized eigenvalues is only $n - 1 = rk(I_{n-1,1})$).

In the case of a skeletal structure (M, U) in $\mathbb{R}^3$ with associated boundary $\mathcal{B}$ and defining region Ω, then at a point $x_0 \in \partial M$, there is a single principal edge curvature, which we denote by κ_E. Then in the case (M, U) locally has a swept representation, we can compute κ_E from the principal relative curvature.

Proposition 2.8 *Suppose the skeletal structure (M, U) in $\mathbb{R}^3$ locally has in a neighborhood of $x_0 \in \partial M$ a swept representation via a family of planes $\{\Pi_t\}_{t \in \partial M}$. Then,*

$$\kappa_E \quad = \quad \kappa_{rel} \qquad (12)$$

We give the proof of this proposition in §5. We also give a simplified method to compute κ_{rel} along edges in Corollary 3.5.

Computing the Shape Operator for Polar Swept Hypersurfaces

Suppose now that $(M, \tilde{M}, U)$ is a swept contracted skeletal structure as in Example 1.7. Then, we let $\mathcal{B}(s) = \psi_s(\tilde{M})$ denote a level set of the radial flow. We may define on $\mathcal{B}(s)$ the vector field U which at $\Psi(x, s)$ is $U(s)(x)$ for $x \in \tilde{M}$. By assumption, for $s < \varepsilon$, this vector field does not lie in any tangent space at a smooth point, nor limiting tangent space at any of the points coming from singular points of M. We can define the radial shape operator $S_{rad\,s}$ for $(\mathcal{B}(s), U(s))$. Because the $(n+1-k)$–subspaces Π_t are transverse to the strata of $\mathcal{B}(s)$ and the limiting tangent spaces at singular points, we can view $\mathcal{B}(s)$ as a swept skeletal structure and give a calculation analogous to Proposition 2.5. This requires computing the radial shape operator for each slice $\mathcal{B}(s)_t$ of $\mathcal{B}(s)$ by Π_t and the relative shape operator for this swept skeletal structure. However, we want to express both of these in terms of $(M, \tilde{M}, U)$.

To define the relative shape operator for $(M, \tilde{M}, U)$, suppose $\tilde{x}_0 \in \tilde{M}$ with $p(\tilde{x}_0) = x_0$ and $x_0 \in \Pi_{t_0}$. Then, for $v \in T_{x_0}M$, with a lift $\tilde{v} \in T_{\tilde{x}_0}\tilde{M}$ we define $S_{rel} : T_{x_0}M \to T_{x_0}M$ by

$$S_{rel,(\tilde{x}_0)}(v) \quad = \quad -\text{proj}_{\Pi_{t_0}}((\frac{\partial U_1}{\partial \tilde{v}}) \qquad (13)$$

As $S_{rel,(\tilde{x}_0)}$ is an operator on $T_{x_0}M$, the "relative feature" is the dependence on $\tilde{x}_0 \in \tilde{M}_{(x_0)} = p^{-1}(x_0)$. As for the relative shape operator for swept skeletal structures, the relative shape operator for $(M, \tilde{M}, U)$ is well–defined.

Lemma 2.9 *S_{rel} is well–defined.*

Proof: It is only necessary to show the definition is independent of the lift $\tilde{v}$. This follows because on $\tilde{M}_{(x_0)} = p^{-1}(x_0)$, U_1 maps to Π_{t_0}, so if $w \in \tilde{T}_{x_0}M_{(x_0)}$, then $\frac{\partial U_1}{\partial \tilde{w}} \in \Pi_{t_0}$. □

We then choose $\mathbf{v} = \{v_1, \ldots, v'_k\}$ for $T_{\tilde{x}_0}\tilde{M}$ which map under $dp_{(\tilde{x})}$ to a basis for $T_{x_0}M$. We also choose a basis $\mathbf{v}' = \{v'_1, \ldots, v'_{n-k}\}$ for $T_{x_0}\tilde{M}_{t_0}$. Together $\mathbf{v}'$ and $\mathbf{v}$ give us a basis $\mathbf{v}''$ for $T_{\tilde{x}_0}\tilde{M}$. Under $d\psi_{s\,(\tilde{x}_0)}$, $\mathbf{v}''$ maps to a basis $\mathbf{v}'''$ for $T_{y_0}\mathcal{B}(s)$ where $\psi_s(\tilde{x}_0) = y_0$.

Proposition 2.10 *The matrix representation of S_{rad} for $\mathcal{B}(s)$ at the point y_0 with respect to the basis $\mathbf{v}'''$ is given by*

$$S_{rad,s,\mathbf{v}'''} = \qquad (14)$$
$$\begin{pmatrix} -\frac{1}{sr} \cdot I_{n-k} & * \\ 0 & S_{rel,(\tilde{x}_0)\,\mathbf{v}} \cdot (I - srS_{rel,(\tilde{x}_0)\,(\tilde{x}_0)\,\mathbf{v}})^{-1} \end{pmatrix}$$

We obtain the following corollary for polar swept surfaces, which are just generalized tubes along a curve.

Corollary 2.11 *Suppose $(M, \tilde{M}, U)$ defines a polar swept surface in $\mathbb{R}^3$ with notation as above. Then, the matrix representation of S_{rad} for $\mathcal{B}(s)$ at the point y_0 with respect to the basis $\mathbf{v}'''$ is given by*

$$S_{rad,s,\mathbf{v}'''} = \begin{pmatrix} -\frac{1}{sr} & * \\ 0 & \frac{\kappa_{rel}}{(1 - sr\kappa_{rel})} \end{pmatrix} \qquad (15)$$

where κ_{rel} is evaluated at the point $\tilde{x}_0 = (t, \theta)$ corresponding to y_0 under the map ψ_s.

Example 2.12 *We consider the special case of a polar swept surface defined for $\gamma(t)$ a unit speed curve, with the planes Π_t normal to $\gamma(t)$. By Corollary 2.11, to explicitly give the matrix representation for S_{rad} for $\mathcal{B}(s)$ in this case, it remains only to compute the term $*$ in the upper high hand corner. We represent $U_1 = \cos(\theta)e_1 + \sin(\theta)e_2$ and denote the orthogonal complement $U_\theta = -\sin(\theta)e_1 + \cos(\theta)e_2$. Then, we may write*

$$\frac{\partial U_1}{\partial t} = \beta_1 U_1 + \beta_2 U_\theta - \kappa_{rel}\mathbf{T}.$$

Then, a straightforward calculation shows the upper right hand entry $*$ is given by

$$-\frac{\beta_2}{sr(1 - sr\kappa_{rel})}$$

In the special case where $e_1 = \mathbf{N}$ and $e_2 = \mathbf{B}$, a direct calculation with the Frenet formulas shows $\beta_1 = 0$, $\beta_2 = \tau$ (the torsion of $\gamma(t)$) and

$$\kappa_{rel} = \kappa\cos\theta \qquad (16)$$

(with κ denoting the usual differential geometric curvature). Then, the upper right hand entry $*$ is given by

$$-\frac{\tau}{sr(1 - sr\kappa\cos\theta)}$$

Remark *For a special class of tubes considered by Mike Kerchove (unpublished) where internal spheres on the central curve are tangent to the boundary along a circle, the circles lie in a family of planes along the curve of circle centers. This defines a special type of swept polar surface, and the above formulas recover his computations of the radial shape operator but in terms of the swept representation.*

3. Relative Principal Curvature Conditions Implying the Smoothness of the Boundary

In this section we derive conditions that a surface in $\mathbb{R}^3$ locally formed as a swept surface from a family of smooth planar curves is itself smooth. Figure 5 illustrates how this may fail.

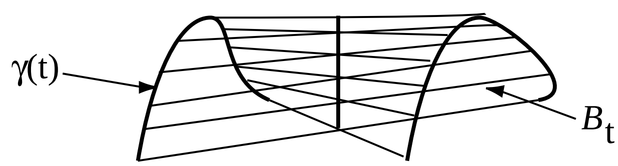

Fig. 5. Failure of smoothness for a surface swept by a smooth family of planar curves $\mathcal{B}_t$ along the smooth space curve $\gamma(t)$. In this case $\mathcal{B}_t$ is a family of straight lines.

We consider the case that (M, U) is locally a swept skeletal structure by a smooth family of planes $\{\Pi_t\}$ for which the skeletal structures (M_t, U_t) have smooth associated boundary curves $\mathcal{B}_t$ in Π_t (see figure 6). We allow points of the open set W where we have the swept representation to be smooth points, edge points, or general singular points; however, we suppose that the Π_t are transverse to the curves of singular points in M such as Y-junction curves and edge curves. Also, at codimension 2 singular points such as fin points and 6–junction points, the Π_t are also transverse to the limiting tangent planes of the regular points and the limiting tangent lines from the $Y-$ junction curves.

Two simple examples where such swept representations are relevant are along edge curves of medial axes, as in Example 3.6, or for generalized offset surfaces, Example 3.8. Then, we give conditions which ensure that the associated boundary $\mathcal{B}$ of (M, U) is smooth by using the conditions from [D1] (alternately see [D3] or [PS, Chap. 3]). The three conditions which ensure smoothness are stated in terms of the principal radial curvatures and the edge curvatures, and a compatibility condition [D1, Theorem 2.5].

Modeling with Swept Surfaces: Conditions Implying Smoothness

We suppose that for each t the swept skeletal structures (M_t, U_t) in Π_t satisfy the three conditions and that the associated boundary curves are smooth (see e.g. Figures 6 and 7) . This assumption implies that the principal radial curvature κ_{rt} for each curve $\mathcal{B}_t$ satisfies the following condition.
(*Radial Curvature Condition*) For all points of each M_t not on ∂M_t (which are the end points of M_t)

$$r < \frac{1}{\kappa_{rt}} \quad \text{if } \kappa_{rt} > 0$$

Then, the condition for smoothness is the following.
Theorem 3.1 *Let (M, U) denote the locally swept skeletal structure on an open set W of M, with the associated (M_t, U_t) having smooth associated boundary curves $\mathcal{B}_t$ satisfying the radial curvature conditions. If at all points of W, (M, U) satisfies the relative curvature condition :*

$$r < \frac{1}{\kappa_{rel}} \quad \text{if } \kappa_{rel} > 0$$

then the associated boundary $\mathcal{B}$ of (M, U) will be smooth at all points of $\mathcal{B}$ corresponding to the points of W.

Proof: By our assumption on κ_{rt} and Corollary 2.7 and Proposition 2.8, we have

(i) (*Radial Curvature Condition*) For all points of each M_t off ∂M_t

$$r \ < \ \min\{\frac{1}{\bar{\kappa}}\} \quad \text{for } \bar{\kappa} \text{ from among those } \kappa_{rt} \text{ or } \kappa_{rel} \text{ which are} > 0$$

(ii) (*Edge Condition*) For all points of $\overline{\partial M_t}$ (closure of ∂M_t)

$$r < \frac{1}{\kappa_E} (= \frac{1}{\kappa_{rel}}) \qquad \text{if } \kappa_{rel} > 0$$

These conditions imply that no singularities are formed by the radial flow from the smooth points, and new singularities are not created for the flow from the singular points of M. It remains to see that at images of singular points and edge points we have well–defined tangent planes. This is usually checked using the compatibility condition in Theorem 2.5 of [D1]. However, by assumption, the curves $\mathcal{B}_t$ are smooth at branch points or end points of M_t. Hence, they are smooth in the slices by the Π_t which are transverse to the strata of M. Thus, from each direction, the tangent plane at a point of $\mathcal{B}$ is formed from the tangent line corresponding to the curve coming from the curve in M and the tangent line for the transverse curve $\mathcal{B}_t$. Hence, the tangent plane is unique. This completes the proof. □

We also give an analogue of Theorem 3.1 for polar swept hypersurfaces.
Corollary 3.2 *Let $(M, \tilde{M}, U)$ define a polar swept hypersurface in $\mathbb{R}^{n+1}$ with $\dim M = k$ and with notation as above. Suppose for $x_0 \in M$ and $x_0 \in \Pi_{t_0}$,*

$$r(\tilde{x}_0) \ < \ \min\{\frac{1}{\kappa_{rel,i}(\tilde{x}_0)}\} \quad \textit{for } \kappa_{rel,i}(\tilde{x}_0) > 0 \quad (17)$$

for all $\tilde{x}_0 \in \tilde{M}_{t_0}$. Then, the level surfaces of the flow $\mathcal{B}(s)$ will be smooth at points of $\psi_s(\tilde{M}_{t_0})$ for all $0 < s \leq 1$,

Proof: We already know the result holds by assumption for $s < \varepsilon$. Choose one such s. Because we can view the radial flow from $\tilde{M}$ at time $s + s'$ as the radial flow from $\mathcal{B}(s)$ at time s', we can apply the criteria for smoothness of associated boundaries given in [D1, Theorem 2.5] to obtain that the radial flow from $y_0 \in \mathcal{B}(s)$ will be smooth for $0 < s' \leq 1-s$ provided

$$r - sr \ < \ \min\{\frac{1}{\kappa_{r,i}}\} \quad \text{for } \kappa_{r,i} > 0 \qquad (18)$$

where $\{\kappa_{r,i}\}$ are the principal radial curvatures for $(\mathcal{B}(s), U(s))$ at y_0. By Proposition 2.10, these are $-\frac{1}{sr}$, with multiplicity $n - k$, and $\kappa_{rel,i} \cdot (1 - sr\kappa_{rel,i})^{-1}$ where the $\kappa_{rel,i}$ are the principal relative curvatures of $S_{rel,(\tilde{x}_0)}$. As $-\frac{1}{sr} < 0$, (18) reduces to

$$r - sr \ < \ \left(\frac{\kappa_{rel,i}}{(1 - sr\kappa_{rel,i})}\right)^{-1} \quad \text{for } \frac{\kappa_{rel,i}}{(1 - sr\kappa_{rel,i})} > 0 \quad (19)$$

However, (17) implies that $\kappa_{rel,i}$ has the same sign as $\kappa_{rel,i} \cdot (1 - sr\kappa_{rel,i})^{-1}$. Thus, (19) need only be verified for $\kappa_{rel,i} > 0$. Then, a direct calculation easily shows that (17) with $\kappa_{rel,i} > 0$ implies (19), as required. □

Remark 3.3 *In the case of polar swept surfaces, the condition (17) becomes*

$$r(\theta, t) \ < \ \frac{1}{\kappa_{rel}(\theta, t)} \quad \textit{when } \kappa_{rel}(\theta, t) > 0 \qquad (20)$$

for all (θ, t). This is the exact analogue of Theorem 3.1.

Now we explain how to explicitly compute the principal relative curvature for swept surfaces in $\mathbb{R}^3$.

Computing the Principal Relative Curvature for Swept Surfaces in $\mathbb{R}^3$

To actually compute the relative principal curvature, we give a method in terms of the swept parametrization for (M, U). We suppose that along a curve $\gamma(t)$ in M, we have chosen an orthonormal frame $\{e_1, e_2, e_3\}$ so that the unit radial vector field $U_1 = e_1$ and with $\{e_1, e_2\}$ spanning the plane Π_t through $\gamma(t)$. Then, we represent the curves $\mathcal{B}_t$ parametrized by

$$X(t,\theta) \;=\; \gamma(t) \,+\, c_1(t,\theta)e_1 \,+\, c_2(t,\theta)e_2 \qquad (21)$$

and

$$U(t,\theta) \;=\; \alpha_1(t,\theta)e_1 \,+\, \alpha_2(t,\theta)e_2 \qquad (22)$$

Here, for fixed t, θ is the parameter for curves in the plane Π_t. Next, as usual, the derivatives of the frame field along $\gamma(t)$ may be written

$$\frac{\partial e_i}{\partial t} \;=\; \omega_{i1}e_1 \,+\, \omega_{i2}e_2 \,+\, \omega_{i3}e_3 \quad \text{for } i = 1 \ldots 3 \qquad (23)$$

with (ω_{ij}) skew symmetric.

Second, we may also write

$$\gamma'(t) \;=\; \gamma_1 e_1 \,+\, \gamma_2 e_2 \,+ \gamma_3 e_3 \qquad (24)$$

Since $\gamma'(t)$ is complementary to Π_t, $\gamma_3 \neq 0$.

Then, the relative principal curvature is given by the following.

Proposition 3.4 *In the preceding situation, the relative principal curvature may be computed by*

$$\kappa_{rel} \;=\; -\frac{1}{r} \cdot \frac{\alpha_1\omega_{13} + \alpha_2\omega_{23}}{\gamma_3 + c_1\omega_{13} + c_2\omega_{23}} \qquad (25)$$

We note several consequences of the proposition.

First, suppose that $\gamma(t)$ is an edge curve and $X(t, 0) = \gamma(t)$ so X parametrizes a neighborhood of the edge of M using edge coordinates, and $\{e_1, e_2, e_3\}$ is an orthonormal frame along $\gamma(t)$ as above.

Corollary 3.5 *Along an edge curve* $\gamma(t)$ *of* M,

$$\kappa_E \;=\; \kappa_{rel} \;=\; -\frac{\omega_{13}}{\gamma_3} \qquad (26)$$

Proof: As $X(t, 0) = \gamma(t)$, (21) implies $c_1(t, 0) = c_2(t, 0) = 0$. Also, as $U_1 = e_1$ on $\gamma(t)$, (22) implies $\alpha_2(t, 0) = 0$ and $\alpha_1(t, 0) = r$. Then, the RHS of (25) becomes the RHS of (26). Hence, the result follows from Proposition 3.4. □

Example 3.6 (Modeling crest regions of boundaries) *A crest region of a boundary surface corresponds to*

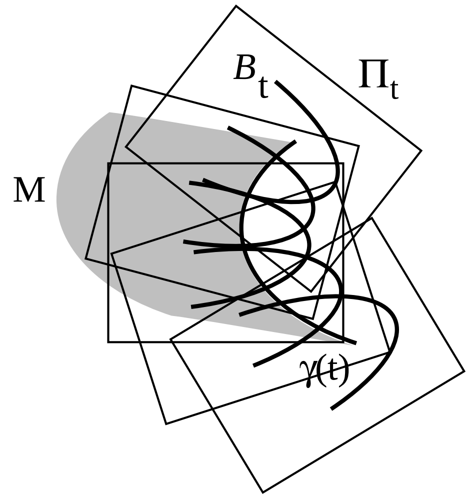

Fig. 6. Swept model of a crest region using a smooth family of ellipses parametrized by the edge curve of the medial axis.

an edge of the medial axis (see e.g. [BGT]). If we would like to model the crest region using a quadratic approximation along the crest curve, one way we can proceed is via a swept surface representation. We suppose that $X(t, \theta)$ *gives edge coordinates for a neighborhood of an edge point, with* $X(t, 0) = \gamma(t)$ *parametrizing the edge. We let* $\{\Pi_t\}$ *be a smooth family of planes transverse to* $\gamma(t)$*, with an orthonormal frame*$\{e_1, e_2\}$ *along* $\gamma(t)$ *for each* Π_t*, such that* $U_1 = e_1$ *along* $\gamma(t)$*. We consider a family of curves* $\mathcal{B}_t \subset \Pi_t$*, whose medial axes are line segments which end at* $\gamma(t)$*, and which together form a neighborhood of the medial axis of the three dimensional region. Then, it will follow from Proposition 2.8 that the principal edge curvature* κ_E*, which controls smoothness of the associated boundary at the crest points, is given by the principal relative curvature* κ_{rel}*. In turn, it is computed without specifying the curves* $\mathcal{B}_t$*. Thus, the edge condition of Theorem 2.5 of [D1] only depends on the values of* r *for the* $\mathcal{B}_t$ *along* $\gamma(t)$*.*

Modeling with families of ellipses

One example is where the $\mathcal{B}_t$ *is a portion of an ellipse* $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$ *with* $a < b$*. Then, the medial axis of the ellipse is the segment on the x-axis* $\left[-\frac{c^2}{b}, \frac{c^2}{b}\right]$*, where* $b^2 = a^2 + c^2$*. If we use the end point* $(\frac{c^2}{b}, 0)$*, then for the parametrization* $(x, y) = (b\cos(\theta), a\sin(\theta))$*, the point on the medial axis is* $(\frac{c^2}{b}\cos(\theta), 0)$*, and* $U = (\frac{a^2}{b}\cos(\theta), a\sin(\theta))$ *(note that here* θ *serves as an edge coordinate for the medial axis). Hence,* $r = \frac{a}{b}(a^2\cos^2(\theta) + b^2\sin^2(\theta))^{\frac{1}{2}}$*, and at the edge point* $r = \frac{a^2}{b}$*. Thus, along the crest curve it is only necessary to ensure that* $\frac{a^2}{b} < \frac{1}{\kappa_{rel}}$

when $\kappa_{rel} > 0$. As a and b are parameters, they can be adjusted to ensure the condition holds.

This will ensure in a small neighborhood of the crest curve that the associated boundary surface is smooth. To ensure that singularities do not develop on a larger region about the crest curve, we use instead the general form for r and verify instead the inequality given by Proposition 3.4 with $(c_1, c_2) = (b\cos(\theta) - \frac{c^2}{b}, a\sin(\theta))$.

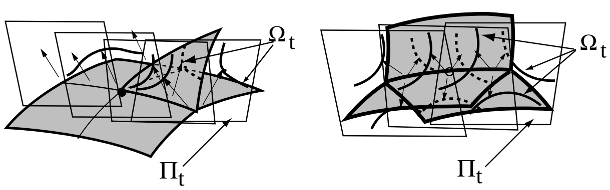

Fig. 7. Modeling a region of a surface corresponding to singular points of the medial axis such as fin points or along Y–junction curves by a swept skeletal structure with a family of smooth curves.

Remark 3.7 *There are other possibilities for modeling crest regions of surfaces with other families of curves depending on parameters such as parabolas. Such modeling has been applied in [HPD]. Also, in the case of medial axes, we can likewise use swept representations for modeling along singular sets of the medial axis such as the Y–junction curves or near fin points or 6– junction points as in Fig. 7. In [TG], such modeling has been carried out.*

Example 3.8 (Modeling with generalized offset surfaces) *A special case of a skeletal structure is the case of a smooth surface M with a radial vector field U. Then, the resulting associated boundary surface $\mathcal{B}$ can be viewed as a generalized offset surface. Then, we can view modeling such an offset surface as being obtained from a swept family of generalized offset curves. The condition that the individual generalized offset curves are smooth is given by the radial curvature condition in Theorem 2.5 of [D1]. Even though the offset curves are smooth it is still possible for the generalized offset surface to have singularities. The condition that the resulting swept generalized offset surface is smooth is given by the same radial curvature condition which reduces to a condition on the principal relative curvature given by Theorem 3.1. Preliminary results obtained for modeling with generalized offset surfaces are given in [C].*

4. Integrals over Swept Regions via Skeletal Integrals

In this section we consider volumetric properties of swept regions. Using the integral formulas from [D4] combined with the results of the earlier sections, we express integrals on the swept region Ω defined by a swept skeletal structure (M_t, U_t) as iterated integrals of skeletal integrals on the slices Ω_t, then integrated over the oriented parameter manifold Γ. Specifically we use the notation of §1, so $\{\Pi_t : t \in \Gamma\}$ is a smooth family of $(n-k+1)$–dimensional affine spaces over a submanifold $\Gamma \subset \mathbb{R}^{n+1}$. We suppose that (M, U) is a swept skeletal structure via the family $\{\Pi_t\}$. We can define a projection $\pi : M \to \Gamma$ by $x \mapsto t$ where $x \in \Pi_t$. For simplicity we assume that both E and Γ are orientable which gives the usual orientation on $\mathbb{R}^{n+1}$ via the diffeomorphism γ.

There is one additional condition which we require to perform volumetric computations. We assume this condition holds throughout this section.

4.1 (Volumetric Condition for Swept Regions) *For a swept skeletal structure (M_t, U_t) for the family of subspaces $\{\Pi_t : t \in \Gamma\}$, defining the region Ω, we require:*

$\{x \in M : M$ *is not transverse to* Π_t *at* $x \in M \cap \Pi_t\}$ *has measure zero in* M.

By $A \subset M$ having measure zero, we mean $A \cap M_{reg}$ has measure zero in M_{reg}.

We begin by giving a "skeletal integral representation" for the integral over Ω of a Borel integrable function $g : \Omega \to \mathbb{R}$. We let $g_1(x, s) = g(x + sU(x))$ for $x \in M$ and $U(x)$ a value of U at x. Then, we define

$$\tilde{g}(x) = \int_0^1 g_1(x, s) \cdot \det(I - srS_{rel}) \cdot \det(I - srS_{rad}(M_t))\, ds. \qquad (27)$$

Theorem 4.2 *Let (M, U) be a swept skeletal structure via the smooth family $\{\Pi_t : t \in \Gamma\}$ which defines the region Ω with smooth boundary $\mathcal{B}$. For a Borel integrable function $g : \Omega \to \mathbb{R}$, we may express the integral*

$$\int_\Omega g\, dV \quad = \quad \int_{\tilde{M}} r \cdot \tilde{g}(x)\ dM. \qquad (28)$$

We recall that the integral on the RHS is over $\tilde{M}$, which means that we integrate over both sides of M (see [D4]).

The proof of Theorem 4.2 follows by applying Theorem 6 of [D4] while using (11) of Corollary 2.7.

We will further represent the integral on the RHS of (28) as iterated integrals first over M_t, and then integrated over Γ. For example, in the special case that the family of subspaces $\{\Pi_t\}$ are parallel and Γ is a linear space orthogonal to the Π_t, then we are reduced to Fubini's theorem, where the integral over each slice Ω_t is given as a skeletal integral. However, in general there are three varying features that each contribute to a modfication: i) the rotational movement of the subspaces Π_t as we vary $t \in \Gamma$, ii) the variation of T_xM with respect to Π_t and $T_{\pi(x)}\Gamma$; and iii) the position of U relative to the skeletal sets M_t and M. All of these variations except the first depend on the point $x \in M_t$. The integral formula we shall give will take into account all three of these variations. For example, even if the subspaces Π_t are parallel, there are still the other two variations to take into account.

Invariants Associated to Swept Skeletal Structures

If Π_t is transverse to M at a point x, and U is a smooth value at x, then we define an invariant ν as follows. Let $\mathbf{n}$ be a unit normal vector to M and on the same side of M as the value of U; and let $\mathbf{n}_1$ be a unit normal vector to M_t in Π_t and on the same side as U. Let $\{v'_1, \dots, v'_{n-k}\}$ be an orthonormal basis for T_xM_t such that $\{\mathbf{n}_1, v'_1, \dots, v'_{n-k}\}$ has positive orientation in Π_t. Let N_x denote the orthogonal complement to T_xM_t in T_xM. For an orthonormal basis $\{v_1, \dots, v_k\}$ for $T_{\pi(x)}\Gamma$, we choose $\{\tilde{v}_1, \dots, \tilde{v}_k\}$ in N_x which map to $\{v_1, \dots, v_k\}$ under $d\pi_x$. These are unique by dimension considerations and the transversality of Π to M. Reordering the v_i if necessary, we suppose $\{\mathbf{n}, v'_1, \dots, v'_{n-k}, \tilde{v}_1, \dots, \tilde{v}_k\}$ has positive orientation for dV, the volume form on M corresponding to $\mathbf{n}$. Then we let

$$\begin{aligned} \nu(x) &= dV(v'_1, \dots, v'_{n-k}, \tilde{v}_1, \dots, \tilde{v}_k) \\ &= \det(\mathbf{n}, v'_1, \dots, v'_{n-k}, \tilde{v}_1, \dots, \tilde{v}_k) \end{aligned} \quad (29)$$

This is independent of the choice of the orthonormal bases having positive orientations. It can also be viewed as the determinant of the matrix of $\pi|N_x$ with respect to orthonormal bases $\{v''_1, \dots, v''_k\}$ for N_x and $\{v_1, \dots, v_k\}$ for $T_t\Gamma$ (with the correct orientation). Thus, it measures the relative position of N_x versus $T_{\pi(x)}\Gamma$, which deals with ii) above.

We define a second invariant at such points

$$\tilde{\rho}(x) = \frac{\rho}{\rho_1}. \quad (30)$$

Here, as in [D4], for a skeletal structure (M, U), $\rho = U_1 \cdot \mathbf{n}$, where $\mathbf{n}$ is the unit normal to M which points on the same side of M as U_1. Similarly we define ρ_1 for M_t, using instead $\mathbf{n}_1$, the unit normal vector to T_xM_t in Π_t. Then, $\tilde{\rho}(x)$ measures the variation iii) above. We give a bound for $\tilde{\rho}$ as a result of the next lemma.

Lemma 4.3 *Suppose M is a hyperplane in $\mathbb{R}^{n+1}$. Let Π be an $(n-k+1)$–subspace transverse to M and let $M' = M \cap \Pi$. Let $\mathbf{n}$ be the unit normal vector to M, and $\mathbf{n}_1$ the unit normal vector to M' in Π. If $U_1 \in \Pi$ is a unit vector then,*

$$U_1 \cdot \mathbf{n} \leq U_1 \cdot \mathbf{n}_1$$

Proof: We may write $\mathbf{n}_1 = a\mathbf{n} + w$, where $w \in M$. Since both $\mathbf{n}_1$ and $\mathbf{n}$ are orthogonal to M', so is w. As $\mathbf{n}_1$ is a unit vector, $|a| \leq 1$. As $U_1 \in \Pi$, we may also write $U_1 = c\mathbf{n}_1 + v$ with $v \in M'$. Then, on the one hand, from the previous representation for U_1,

$$U_1 \cdot \mathbf{n}_1 = c$$

Second, first using the representation for U_1 and then that for $\mathbf{n}_1$

$$U_1 \cdot \mathbf{n} = c\mathbf{n}_1 \cdot \mathbf{n} = ca.$$

Together these yield the result. $\square$

Remark 4.4 *If we apply Lemma 4.3 to T_xM and $M' = T_xM_t$, we obtain $\rho \leq \rho_1$, which implies the bound $0 \leq \tilde{\rho}(x) \leq 1$. We also note that if $\Pi_t \perp M$ at all points of M and for all t, then $\mathbf{n} = \mathbf{n}_1$, so $\rho = \rho_1$ and $\tilde{\rho} \equiv 1$.*

Also, in the special case that Γ is the smooth part of M, then $\nu \equiv 1$.

Expansion as an Iterated Integral

Then, we can expand the integral on the RHS of (28) as an iterated integral of a skeletal integral over M_t, and then integrating over $t \in \Gamma$. By our earlier discussion, there is an open dense subset $\Gamma_0 \subset \Gamma$ whose complement has measure zero, so that Π_t is tranverse to M_t and $\mathcal{B}_t$, and $\Omega_t = \Omega \cap \Pi_t$ is a smooth manifold with boundary $\mathcal{B}_t$. We let $\Omega_0 = \cup_{t \in \Gamma_0} \Omega_t$, and $M_0 = \cup_{t \in \Gamma_0} M_t$. Both are open dense subsets whose complements in their respective spaces have measure zero. Then, the relative shape operator is defined at all points of M_0 and integrals over Ω are the same as integrals over Ω_0.

This time we define for $x \in \Omega_0$ and $g_1(x, s) = g(x + sU(x))$,

$$\bar{g}(x) \;=\; \int_0^1 g_1 \cdot \det(I - srS_{rel}) \cdot \det(I - srS_{rad}(M_t))\, ds. \qquad (31)$$

Then, $\bar{g}$ is Borel measurable on a Borel set Ω_0 whose complement has measure zero. Thus, its integral over Ω is defined.

Theorem 4.5 (Iterated Skeletal Integrals) *Let (M, U) be a swept skeletal structure via the smooth family $\{\Pi_t : t \in \Gamma\}$ which defines the region Ω with smooth boundary $\mathcal{B}$. For Borel integrable function $g : \Omega \to \mathbb{R}$, we may express the integral as an iterated integral*

$$\int_\Omega g\, dV \;=\; \int_\Gamma \int_{\tilde{M}_t} \bar{g}(x) \cdot r\; d\bar{M}_t\, dV_\Gamma. \qquad (32)$$

Here $d\bar{M}_t$ is a *relative medial measure*

$$d\bar{M}_t \;=\; \nu \cdot \tilde{\rho}\; dM_t \;=\; \nu \cdot \rho\; dA_t$$

where $dM_t = \rho_1 \cdot dA_t$ is the medial measure on M_t, with dA_t the Riemannian volume measure on M_t.

The proof of Theorem 4.5 will be given in §7. We next derive several consequences of this theorem.

We may use (32) and Theorem 6 of [D4] (applied to M_t) to rewrite the integral over Ω as an iterated integral over Ω_t and then over Γ.

Corollary 4.6 *In the preceding situation of Theorem 4.5, the integral of g over Ω may be expressed as an iterated integral over the regions Ω_t.*

$$\int_\Omega g\, dV \;=\; \int_\Gamma \int_{\Omega_t} g(x) \cdot \nu \cdot \tilde{\rho} \cdot det(I - srS_{rel})\; dV_t\, dV_\Gamma. \qquad (33)$$

Remark 4.7 *In the case that $\Omega \subset \mathbb{R}^3$, in the preceding integrals $S_{rad}(M_t)$ and S_{rel} are multiplication by the scalars $\kappa_{r\,t}$, resp. κ_{rel}, and the determinants in (31) are just the factors $(1 - sr\kappa_{r\,t})$, resp. $(1 - sr\kappa_{rel})$.* As a consequence of (33), we see that the $(n+1)$–dimensional volume of Ω (which is given by the integral of $g \equiv 1$ over Ω) is not obtained by integrating the $(n-k+1)$–dimensional volume of Ω_t over Γ with an appropriate integrating factor; but instead, by the integral of $\nu \cdot \tilde{\rho} \cdot \det(I - srS_{rel})$ over Ω_t, and then integrated over Γ. For example, for a swept region $\Omega \subset \mathbb{R}^3$

Corollary 4.8 *For a swept region $\Omega \subset \mathbb{R}^3$ along a curve $\gamma(t)$,*

$$vol(\Omega) \;=\; \int_\gamma \int_{\Omega_t} \nu \cdot \tilde{\rho} \cdot (1 - sr\kappa_{rel})\; dA\, ds. \qquad (34)$$

In the case that we want to integrate g over a subregion $\Delta \subset \Omega$, we may apply the Crofton-type formula from [D4] to express $\int_\Delta g$ as an iterated integral. Such a formula computes integrals over the region Δ by first integrating over the intersection of the region with radial lines and then integrating the resulting function over the skeletal set M which parametrizes such lines.

We let

$$\bar{g}_\Delta(x) \;=\; \int_0^1 \chi_\Delta \cdot g_1 \cdot \det(I - srS_{rel}) \cdot \det(I - srS_{rad}(M_t))\, ds. \qquad (35)$$

where χ_Δ is the characteristic function of Δ.

Theorem 4.9 (Iterated Skeletal Crofton-Type Formula) *Suppose (M, U) is a swept skeletal structure which defines a region Ω. Let $\Delta \subset \Omega$ be Borel measurable and let $g : \Delta \to \mathbb{R}$ be Borel measurable and integrable. Then, $\bar{g}$ is defined for almost all $U(x)$; it is integrable on $\tilde{M}$; and*

$$\int_\Delta g\, dV \;=\; \int_\Gamma \int_{\tilde{M}_t} r \cdot \bar{g}_\Delta(x)\; d\bar{M}_t\, dV_\Gamma. \qquad (36)$$

Note that $\tilde{g}_\Gamma$ will vanish for all $(x, U(x))$ for which the radial line $\{x + tU(x) : 0 \le t \le 1\}$ only intersects Γ in a set of measure 0.

Next we expand the integral in (32) in terms of moment integrals on Γ of radial moments of g.

Expansion by Moment Integrals

As in [D4], we can expand the determinants in the integrals in Theorems 4.5 and 4.9 and express these integrals in terms of moment integrals. For example, in [TG] and [T], moment integrals are used to compare shape fit for matching.

At a point $x \in M$ where Π_t is transverse to M, we have the relative shape operator S_{rel} defined with principal relative curvatures $\{\kappa_{rel,i}\}$. We let $\sigma_{rel,j}$ denote the j–th elementary symmetric function in the $\kappa_{rel,i}$. (so e.g. $\sigma_{rel,1} = \mathrm{tr}\,(S_{rel})$, $\sigma_{rel,k} = \det(S_{rel})$, etc). These invariants are measures of the variation in i) above.

By our earlier discussion, the relative shape operator is defined at all points of open dense subset $M_0 \subset M$, whose complement has measure zero. Hence, the $\sigma_{rel,j}$ are smooth on M_0, so Borel measurable on M.

Now, we may then state a formula for the integral of $g : \Omega \to \mathbb{R}$ over Ω. We define for $x \in M_0$ with $x \in \Pi_t$ and non-negative integer j, the *j–th radial moment of g* for the slice M_t

$$m_j(g)(x) \;=\; \int_0^1 g_1(x,s) \cdot s^j \cdot \det(I - srS_{rad}(M_t))\, ds. \quad (37)$$

where $g_1(x,s) = g(x + sU(x))$. In the special case of $j = 0$, we obtain a special type of weighted average along a radial line

$$m_0(g)(x) \;=\; \tilde{g}(x) \;=\; \int_0^1 g_1(x,s) \cdot \det(I - srS_{rad}(M_t))\, ds. \quad (38)$$

Next, we define a relative skeletal moment integral over M_t.

$$I_{rel,j+1}(h)(x) \;=\; \int_{\tilde{M}_t} h(x) \cdot r^{j+1} \cdot \sigma_{rel,j}\; d\bar{M}_t. \quad (39)$$

Then, we finally can give a skeletal integral representation for the integral of g over Ω

Theorem 4.10 *Let (M, U) be a swept skeletal structure via the smooth family $\{\Pi_t : t \in \Gamma\}$ which defines the region Ω with smooth boundary $\mathcal{B}$. For Borel integrable function $g : \Omega \to \mathbb{R}$,we may express the integral*

$$\int_\Omega g\, dV \;=\; \sum_{j=0}^{k} (-1)^j \int_\Gamma I_{rel,j+1}(m_j(g))\; dV_\Gamma. \quad (40)$$

As a corollary, we consider the case of a swept skeletal structure (M, U) in $\mathbb{R}^3$ via the smooth family $\{\Pi_t\}$ on a curve parametrized by $\gamma(t)$, which defines a region $\Omega \subset \mathbb{R}^3$ with smooth boundary $\mathcal{B}$. Then, the relative shape operator is just multiplication by the principal relative curvature κ_{rel}. We obtain

Corollary 4.11 *Let (M, U) be a swept skeletal structure in $\mathbb{R}^3$ via the smooth family of planes $\{\Pi_t\}$ on a curve parametrized by $\gamma(t)$, which defines a region $\Omega \subset \mathbb{R}^3$ with smooth boundary $\mathcal{B}$. For Borel integrable function $g : \Omega \to \mathbb{R}$, we may express the integral*

$$\int_\Omega g\, dV \;=\; \int_\gamma (I_{rel,1}(m_0(g)) - I_{rel,2}(m_1(g)))\, ds \quad (41)$$

Integrals over Regions Bounded by Polar Swept Hypersurfaces

Finally, we give alternate forms of these theorems for the case of a region Ω bounded by a polar swept hypersurface $\mathcal{B}$ defined via the smooth family of $n - k + 1$–dimensional affine planes $\{\Pi_x\}$ parametrized by $x \in M$. The polar swept structure is defined by a map $\psi : M \times \mathbb{R}^{n-k+1} \to \mathbb{R}^{n+1}$, where for each $x \in M$, $\psi_x = \psi(x, \cdot)$ maps $\{x\} \times \mathbb{R}^{n-k+1}$ isometrically to Π_x. Via this identification, locally the unit sphere bundle $\tilde{M} \simeq M \times S^{n-k}$, and the unit radial vector field $U_1 : \tilde{M} \to \mathbb{R}^{n+1}$ maps to the standard unit radial vector field at the origin of Π_x.

In this case, we only use a variant of the invariant ν. We let $\{v'_1, \dots, v'_{n-k+1}\}$ be an orthonormal basis for Π_x, and $\{v_1, \dots, v_k\}$, an orthonormal basis for T_xM so that $\{v'_1, \dots, v'_{n-k+1}, v_1, \dots, v_k\}$ has positive orientation for $\mathbb{R}^{n+1}$. Then, for polar swept surfaces we let

$$\begin{aligned}\nu(x) \;&=\; dV(v'_1, \dots, v'_{n-k+1}, \tilde{v}_1, \dots, \tilde{v}_k) \\ &=\; \det(v'_1, \dots, v'_{n-k+1}, \tilde{v}_1, \dots, \tilde{v}_k) \quad (42)\end{aligned}$$

As for ν defined for swept skeletal structures by (29), (42) is independent of the choices of orthonormal bases.

Theorem 4.12 *Let Ω be a swept region bounded by a polar swept hypersurface $\mathcal{B}$ via the smooth family $\{\Pi_x : x \in M\}$. For Borel integrable function $g : \Omega \to \mathbb{R}$, we let $g_1(s, x, \theta) = g(\psi(x, \theta) + sU(x, \theta))$. Then, we may express the integral as an iterated integral*

$$\int_\Omega g\, dV \;=\; \int_M \int_{S^{n-k}} \int_0^1 g_1(s, x, \theta) \cdot s^{n-k} \cdot \det(I - srS_{rel}) r^{n-k+1}\, ds\, dS\, d\bar{M} \quad (43)$$

where dS is the volume form on S^{n-k} and $d\bar{M} = \nu \cdot dA$, for dA the Riemannian volume form on M.

The proof of Theorem 4.12 will be given in §7.

In the case of a swept region $\Omega \subset \mathbb{R}^3$ bounded by a polar swept surface along a cuve $\gamma(t)$, the formula takes the following form (rewritten so r becomes a limit of integration).

Corollary 4.13 *Let Ω be a swept region $\Omega \subset \mathbb{R}^3$ bounded by a polar swept surface $\mathcal{B}$ along a curve $\gamma(t)$. For Borel integrable function $g : \Omega \to \mathbb{R}$,we may express the integral as an iterated integral*

$$\int_\Omega g\, dV \;= \quad (44)$$

$$\int_{\gamma}\int_{S^1}\left(\int_0^r g(s',t,\theta)\cdot s'\cdot(1-s'\kappa_{rel})\,ds'\right)d\ell\,d\bar{s}$$

where $d\ell$ is the length form on S^1 and $d\bar{s} = \nu\cdot ds$, for ds the length form on $\gamma(t)$.

5. Proofs of Propositions 2.8 and 3.4

Both Propositions 2.8 and 3.4 involve swept skeletal structures in $\mathbb{R}^3$ defined by a family of planes $\{\Pi_t\}$ along a curve $\gamma(t)$. We use the notion for Proposition 3.4, and first prove that proposition by expliciting computing the relative shape operator. Second, although there is probably a more elegant way to prove Proposition 2.8, we shall proceed directly and use a formula for the principal edge curvature for skeletal structures in $\mathbb{R}^3$ given in [D3], and see that the computation yields exactly (26).

Proof of Proposition 3.4 : In terms of the notation for this proposition, we begin by representing the basis $\{U, X_\theta, X_t\}$ in terms of the orthonormal frame $\{e_1, e_2, e_3\}$. Here X_θ and X_t denote partial derivatives with respect to θ and t.

$$X_\theta \;=\; c_{1\,\theta}e_1 + c_{2\,\theta}e_2 \qquad (45)$$

where we abbreviate $\frac{\partial c_i}{\partial\theta} = c_{i\,\theta}$ and $\frac{\partial c_i}{\partial t} = c_{i\,t}$. Also,

$$X_t \;=\; \gamma'(t) + c_{1\,t}e_1 + c_{2\,t}e_2 + c_1\frac{\partial e_1}{\partial t} + c_2\frac{\partial e_2}{\partial t} \qquad (46)$$

From (46), and (23) and the skew symmetry of ω_{ij}, we obtain

$$X_t \;=\; (\gamma_1 - c_2\omega_{12} + c_{1\,t})e_1 + (\gamma_2 + c_1\omega_{12} + c_{2\,t})e_2 + (\gamma_3 + c_1\omega_{13} + c_2\omega_{23})e_3 \qquad (47)$$

We denote the coefficient of each e_i in (47) by $\tilde{\gamma}_i$.

Then, the matrix for the representation of $\{U, X_\theta, X_t\}$ in terms of the orthonormal frame $\{e_1, e_2, e_3\}$ is given by

$$\mathbf{A} \;=\; \begin{pmatrix} \alpha_1 & c_{1\,\theta} & \tilde{\gamma}_1 \\ \alpha_2 & c_{2\,\theta} & \tilde{\gamma}_2 \\ 0 & 0 & \tilde{\gamma}_3 \end{pmatrix} \qquad (48)$$

We note that $\mathbf{A}$ has the form

$$\mathbf{A} \;=\; \begin{pmatrix} \mathbf{A}_1 & \mathbf{B} \\ 0 & b_3 \end{pmatrix} \qquad (49)$$

for a 2×2 matrix $\mathbf{A}_1$ and column vector $\mathbf{B}$. Hence, the matrix representing the orthonormal frame $\{e_1, e_2, e_3\}$ with respect to the basis $\{U, X_\theta, X_t\}$ is given by $\mathbf{A}^{-1}$ which has the form

$$\mathbf{A}^{-1} \;=\; \begin{pmatrix} \mathbf{A}_1^{-1} & -b_3^{-1}\mathbf{A}_1^{-1}\mathbf{B} \\ 0 & b_3^{-1} \end{pmatrix} \qquad (50)$$

Next, we compute

$$\frac{\partial U}{\partial t} \;=\; \alpha_{1\,t}e_1 + \alpha_1\frac{\partial e_1}{\partial t} + \alpha_{2\,t}e_2 + \alpha_2\frac{\partial e_2}{\partial t} \qquad (51)$$

where $\alpha_{i\,t}$ denotes $\frac{\partial\alpha_i}{\partial t}$. We can rewrite (51)

$$\frac{\partial U}{\partial t} \;=\; (\alpha_{1\,t} - \alpha_2\omega_{12})e_1 + (\alpha_{2\,t} + \alpha_1\omega_{12})e_2 + (\alpha_1\omega_{13} + \alpha_2\omega_{23})e_3 \qquad (52)$$

We may also compute

$$\frac{\partial U}{\partial t} \;=\; \frac{\partial r}{\partial t}U_1 + r\frac{\partial U_1}{\partial t} \qquad (53)$$

As $U_1 \in \Pi_{t_0}$,

$$-\mathrm{proj}_{\Pi_{t_0}}\left(\frac{\partial U}{\partial t}\right) \;=\; -r\cdot\mathrm{proj}_{\Pi_{t_0}}\left(\frac{\partial U_1}{\partial t}\right) \;=\; r\kappa_{rel}X_t \qquad (54)$$

As $e_1, e_2 \in \Pi_{t_0}$, from (52) we obtain

$$-\mathrm{proj}_{\Pi_{t_0}}\left(\frac{\partial U}{\partial t}\right) \;=\; -(\alpha_1\omega_{13} + \alpha_2\omega_{23})\mathrm{proj}_{\Pi_{t_0}}(e_3) \qquad (55)$$

Finally, we may express e_3 in terms of the basis $\{U, X_\theta, X_t\}$, and obtain from (50) that the coefficient of X_t is $\tilde{\gamma}_3^{-1}$. Then, as $U, X_\theta \in \Pi_{t_0}$, we obtain from (54) and (52)

$$r\kappa_{rel}X_t \;=\; -(\alpha_1\omega_{13} + \alpha_2\omega_{23})\cdot\tilde{\gamma}_3^{-1}X_t \qquad (56)$$

Equating the coefficients of X_t and using the expression for $\tilde{\gamma}_3$ gives the desired result. □

We next turn to the proof of Proposition 2.8.

Proof of Proposition 2.8 : We now assume that $\gamma(t)$ parametrizes a part of the edge curve of the skeletal set M and that again $U_1 = e_1$. Here the parametrization $X(t,\theta)$ as defined by (21) gives edge coordinates for a neighborhood of the edge point $\gamma(t_0) = \psi_1(x)$ so that $X(t,0) = \gamma(t)$ and $\frac{\partial X}{\partial\theta}(t,0) = ce_1$ for $c > 0$.

Suppose we have a matrix representation of the edge shape operator S_E with respect to the bases $\{\gamma'(t_0), e_1\}$ and $\{\gamma'(t_0), \mathbf{n}\}$ given by

$$[S_E] \;=\; \begin{pmatrix} b_1 & b_2 \\ c_{\mathbf{n}\,1} & c_{\mathbf{n}\,2} \end{pmatrix} \qquad (57)$$

Then, we recall from Example 2.4 of [D3] that the principal edge curvature is given by

$$\kappa_E \quad = \quad c_{\mathbf{n}\,2}^{-1} \det([S_E]) \tag{58}$$

To compute a matrix representation for the edge shape operator, we must compute $\frac{\partial U_1}{\partial t}$ and $\frac{\partial U_1}{\partial \theta}$ at $(t, \theta) = (t_0, 0)$. First, since $X(t, 0) = \gamma(t)$, and on $\gamma(t)$, $U_1 = e_1$, we have already computed the first of these derivatives in (23)

$$\frac{\partial U_1}{\partial t}(t_0, 0) \quad = \quad \omega_{12} e_2 \, + \, \omega_{13} e_3 \tag{59}$$

Second, from (22)

$$\frac{\partial U}{\partial \theta} \quad = \quad \alpha_{1\,\theta} e_1 \, + \, \alpha_{2\,\theta} e_2 \tag{60}$$

At $(t_0, 0)$, $U_1 = e_1$, so $\alpha_1 = r$ and $\alpha_2 = 0$. Also, for fixed $t = t_0$, $U_1 \in \Pi_{t_0}$ for all θ; hence, $\frac{\partial U_1}{\partial \theta} \in \Pi_{t_0}$ As $\|U_1\| = 1$, $\frac{\partial U_1}{\partial \theta}$ is orthogonal to $U_1 = e_1$, when $\theta = 0$. Thus, $\frac{\partial U_1}{\partial \theta}(t_0, 0) = c e_2$ for some c. From the analogue of (53) for derivatives with respect to θ, we obtain at $(t_0, 0)$

$$\alpha_{1\,\theta} e_1 \, + \, \alpha_{2\,\theta} e_2 \quad = \quad r_\theta e_1 \, + \, r c e_2 \tag{61}$$

Then, at $(t_0, 0)$, $r_\theta = \alpha_{1\,\theta}$, and $c = \frac{\alpha_{2\,\theta}}{\alpha_1}$. Hence,

$$\frac{\partial U_1}{\partial \theta}(t_0, 0) \quad = \quad \frac{\alpha_{2\,\theta}}{\alpha_1} \cdot e_2 \tag{62}$$

Next, we compute the matrix representation of $\{e_1, \mathbf{n}, X_t\}$ with respect to the orthonormal frame $\{e_1, e_2, e_3\}$, which we suppose is positively oriented. Here $\mathbf{n}$ is the unit normal vector field on M pointing on the same side of M as U. We can compute $\mathbf{n}$ as the normalized unit vector field obtained from $X_t \times e_1$.

We are interested in the point $(t_0, 0)$. Since $X(t, 0) = \gamma(t)$, $c_1 = c_2 = 0$ and $c_{1\,t} = c_{2\,t} = 0$ at $(t_0, 0)$. Hence, for the form of X_t in (47), $\tilde{\gamma}_i = \gamma_i$, and we obtain

$$\mathbf{n} \quad = \quad \frac{1}{\hat{\gamma}}(\gamma_3 e_2 \, - \, \gamma_2 e_3) \qquad \text{where } \hat{\gamma}^2 = \gamma_2^2 + \gamma_3^2 \tag{63}$$

Thus, we obtain the matrix for the representation of $\{e_1, \mathbf{n}, X_t\}$ in terms of the orthonormal frame $\{e_1, e_2, e_3\}$ is given by

$$\mathbf{C} \quad = \quad \begin{pmatrix} 1 & 0 & \gamma_1 \\ 0 & -\frac{\gamma_3}{\hat{\gamma}} & \gamma_2 \\ 0 & \frac{\gamma_2}{\hat{\gamma}} & \gamma_3 \end{pmatrix} \tag{64}$$

We again note that $\mathbf{C}$ has the form

$$\mathbf{C} \quad = \quad \begin{pmatrix} 1 & \mathbf{D} \\ 0 & \mathbf{C}_1 \end{pmatrix} \tag{65}$$

for a 2×2 matrix $\mathbf{C}_1$ and row vector $\mathbf{D}$. Hence, the matrix representing the orthonormal frame $\{e_1, e_2, e_3\}$ with respect to the basis $\{e_1, \mathbf{n}, X_t\}$ is given by $\mathbf{C}^{-1}$ which has the form

$$\mathbf{C}^{-1} \quad = \quad \begin{pmatrix} 1 & -\mathbf{D}\mathbf{C}_1^{-1} \\ 0 & \mathbf{C}_1^{-1} \end{pmatrix} \tag{66}$$

A straightforward calculation shows (using $\hat{\gamma}^2 = \gamma_2^2 + \gamma_3^2$)

$$\mathbf{C}_1^{-1} \quad = \quad -\frac{1}{\hat{\gamma}} \cdot \begin{pmatrix} \gamma_3 & -\gamma_2 \\ -\frac{\gamma_2}{\hat{\gamma}} & -\frac{\gamma_3}{\hat{\gamma}} \end{pmatrix} \tag{67}$$

Then, S_E is given by minus the projection of $\{\frac{\partial U_1}{\partial t}, \frac{\partial U_1}{\partial \theta}\}$ along U onto the subspace with basis $\{X_t, \mathbf{n}\}$. From (62) and (59), we obtain for the matrix representation of S_E

$$[S_E] \;=\; -\begin{pmatrix} \frac{1}{\hat{\gamma}^2}(\gamma_2\omega_{12} + \gamma_3\omega_{13}) & \frac{\alpha_{2\,\theta}}{\alpha_1} \cdot \frac{\gamma_2}{\hat{\gamma}^2} \\ (-\gamma_3\omega_{12} + \gamma_2\omega_{13}) & -\frac{\alpha_{2\,\theta}}{\alpha_1} \cdot \gamma_3 \end{pmatrix} \tag{68}$$

Hence, applying (58) with $[S_E]$ given by (68), we obtain after expanding and simplifying,

$$\kappa_E \quad = \quad -\frac{\omega_{13}}{\gamma_3} \tag{69}$$

However, this is exactly the formula for κ_{rel} in the case $\gamma(t)$ parametrizes an edge curve given in Corollary 3.5. □

6. Proof of Proposition 2.10

Proof :

To prove Proposition 2.10, we must compute the radial shape operator for $\mathcal{B}_t(s)$ and then the relative shape operator for the swept skeletal structure $(\mathcal{B}(s), U(s))$.

Lemma 6.1 *For the skeletal structure $(\mathcal{B}_t(s), U(s))$,*

$$S_{rad}(\mathcal{B}_t(s)) \quad = \quad -\frac{1}{sr} \cdot I_{n-k}$$

Proof of the Lemma : We express the parametrization of $\mathcal{B}_t$ by $r(\theta)\cdot U_1$, where U_1 is the unit radial vector field in Π_t and $\theta \in S^{n-k}$, the unit sphere in E_x. Then, $\psi(\theta) = sr(\theta)\cdot U_1$ is the parametrization of $\mathcal{B}_t(s)$. We let $(\theta_1,\ldots,\theta_{n-k})$ denote local coordinates for S^{n-k} near $\tilde{x}$.

We compute

$$v_i \;=\; \frac{\partial \psi}{\partial \theta_i} \;=\; s\left(\frac{\partial r}{\partial \theta_i}\cdot U_1 \;+\; r\cdot\frac{\partial U_1}{\partial \theta_i}\right) \tag{70}$$

Let $w_i = \frac{\partial U_1}{\partial \theta_i} \in T_{\tilde{x}}S^{n-k}$. Then,

$$\frac{\partial U_1}{\partial v_i} \;=\; \frac{\partial U_1}{\partial \theta_i} \;=\; w_i$$

Hence, from (70)

$$\frac{\partial U_1}{\partial v_i} \;=\; w_i \;=\; \frac{1}{sr}v_i - \frac{1}{r}\cdot\frac{\partial r}{\partial \theta_i}\cdot U_1$$

Thus,

$$-\mathrm{proj}_U\left(\frac{\partial U_1}{\partial v_i}\right) \quad=\quad -\frac{1}{sr}v_i$$

giving the result. □

Remark *Lemma 6.1 says that the radial shape operator for $\mathcal{B}_t(s)$ contains essentially no information about the hypersurface $\mathcal{B}_t$. However, we note that all principal radial curvatures $= -\frac{1}{sr}$ are negative, so there are no restrictions on the level sets being smooth . This is consistent with Corollary 3.2, as a level set is obtained from $\mathcal{B}_t$ by scalar multiplication.*

To complete the proof of the proposition, we must compute the relative shape operator. Let $y = \psi_s(\tilde{x})$. First, we compute a basis for $T_y\mathcal{B}(s)$. A parametrization of $\mathcal{B}(s)$ is given by

$$\Psi(t,\theta) \quad=\quad X(t) \;+\; sr(t,\theta)\cdot U_1(t,\theta)$$

where $t = (t_1,\ldots,t_k)$ and $\theta = (\theta_1,\ldots,\theta_{n-k})$ are local coordinates for Γ, resp. S^{n-k}, for a local trivialization of E; and X(t) is the local embedding of Γ. Then, we let

$$v_i \;\stackrel{def}{=}\; \frac{\partial \Psi}{\partial \theta_i} \;=\; s\left(\frac{\partial r}{\partial \theta_i}\cdot U_1 \;+\; r\cdot\frac{\partial U_1}{\partial \theta_i}\right) \tag{71}$$

and

$$w_j \;\stackrel{def}{=}\; \frac{\partial \Psi}{\partial t_j} \;=\; \frac{\partial X(t)}{\partial t_j} + s\left(\frac{\partial r}{\partial t_j}\cdot U_1 + r\cdot\frac{\partial U_1}{\partial t_j}\right) \tag{72}$$

where all partials are evaluated at (t_0,θ_0) corresponding to $\tilde{x} \in E_x$.

Then, $\{v_1,\ldots,v_{n-k},w_1,\ldots,w_k\}$ is a basis for $T_y\mathcal{B}(s)$. We let N_y denote the subspace with basis $\{w_1,\ldots,w_k\}$, which is complementary to $T_y\mathcal{B}_t(s)$ in $T_y\mathcal{B}(s)$, which has a basis $\{v_1,\ldots,v_{n-k}\}$. We also let $u_j = \frac{\partial X}{\partial t_j}(t_0)$, so $\{u_1,\ldots,u_k\}$ is a basis for $T_x\Gamma$. From the definition of the relative shape operator

$$\frac{\partial U_1}{\partial u_j} \;=\; \frac{\partial U_1}{\partial t_j} \;=\; z_j \;-\; (S_{rel}^T\cdot \mathbf{u})_j \tag{73}$$

where $\mathbf{u}$ is a column vector with j–th entry the vector u_j. Then, (73) can be more concisely written

$$\frac{\partial U_1}{\partial \mathbf{u}} \quad=\quad \mathbf{z} \;-\; S_{rel}^T\cdot\mathbf{u} \tag{74}$$

where $\frac{\partial U_1}{\partial \mathbf{u}}$ and $\mathbf{z}$ are column vectors with j–th entries $\frac{\partial U_1}{\partial u_j}$, resp. z_j.

Likewise from (72), we obtain the vector equation

$$\mathbf{w} \;=\; \mathbf{u} + s\left(\frac{\partial r}{\partial \mathbf{t}}\cdot U_1 \;+\; r\cdot\frac{\partial U_1}{\partial \mathbf{u}}\right) \tag{75}$$

with $\frac{\partial r}{\partial \mathbf{t}}\cdot U_1$ denoting the column vector whose j–th entry is the vector $\frac{\partial r}{\partial t_j}\cdot U_1$. Using (74) we obtain from (75)

$$\mathbf{w} \;=\; s\frac{\partial r}{\partial \mathbf{t}}\cdot U_1 \;+\; sr\cdot\mathbf{z} \;+\; (I - srS_{rel}^T)\cdot\mathbf{u} \tag{76}$$

The first two terms on the RHS of (76) belong to Π_t (with $x \in \Pi_t$). A calculation analogous to that in [D1, Proposition 4.1] shows that Ψ being a diffeomorphism for $0 < s < \varepsilon$ implies that $(I - srS_{rel}^T)$ is invertible for the same range of values for s. Hence, we may write

$$\mathbf{u} \quad=\quad \tilde{\mathbf{z}} \;+\; (I - srS_{rel}^T)^{-1}\cdot\mathbf{w} \tag{77}$$

where $\tilde{\mathbf{z}} \in \Pi_t$. Because a value of $U(s)$ on $\mathcal{B}(s)$ is the translate of the corresponding value of U on M (for the appropriate θ), we compute

$$\begin{aligned}\frac{\partial U_1}{\partial w_i}_{|y} \;&=\; \frac{\partial U_1\circ\Psi}{\partial t_i}_{|(t_0,\theta_0)} \;=\; \frac{\partial U_1(t,\theta)}{\partial t_i}_{|(t_0,\theta_0)} \\ &=\; \frac{\partial U_1}{\partial u_i}_{|(t_0,\theta_0)}\end{aligned} \tag{78}$$

Applying (74) to (78), and using (77) to represent $\mathbf{u}$ we obtain

$$\begin{aligned}\frac{\partial U_1}{\partial \mathbf{w}} \;&=\; \mathbf{z} \;-\; S_{rel}^T\tilde{\mathbf{z}} \;-\; S_{rel}^T(I - srS_{rel}^T)^{-1}\cdot\mathbf{w} \\ &=\; \tilde{\tilde{\mathbf{z}}} \;-\; (S_{rel}(I - srS_{rel})^{-1})^T\cdot\mathbf{w}\end{aligned} \tag{79}$$

Since the entries of $\tilde{\tilde{\mathbf{z}}} = \mathbf{z} - S_{rel}^T\tilde{\mathbf{z}}$ belong to Π_t, by the definition of the relative shape operator for $(\mathcal{B}(s), U(s))$, (79) implies

$$S_{rel}(\mathcal{B}(s)) \quad=\quad S_{rel}(I - srS_{rel})^{-1}.$$

□

7. Proofs of Skeletal Integral Formulas

As we have already indicated, Theorem 4.2 follows from Theorem 6 of [D4]. We next consider Theorem 4.5

Proof of Theorem 4.5 : Because M is a Whitney stratified set, which can be locally paved by the definition of skeletal structure, we may construct a tubular system for M_{sing} whose union forms an open neighborhood W' of measure $\frac{\varepsilon}{2}$ in M_{reg}. Similarly,

$$\Sigma = \{x \in M : \Pi_t \text{ is not transverse to } M \text{ at } x \text{ for some } t \in \Gamma\}$$

has measure zero by the volumetric condition. Thus, we can also find an open neighborhood W'' of measure $\frac{\varepsilon}{2}$ in M_{reg}. We let $W_0 = W' \cup W''$. Then, for any point $x \in M_{reg} \backslash \Sigma$, the map $p : M \to \Gamma$ is a local submersion, so we can find a neighborhood W_α so that $p : W_\alpha \to V_\alpha (= p(W_\alpha))$ is a trivial fibration (with fiber M_α). Then, we can find a locally finite refinement of $\{W_0\} \cup \{W_\alpha\}_\alpha$ and a subordinate partition of unity $\{\chi_0\} \cup \{\chi_\alpha\}_\alpha$. We can pull-back these to the double $\tilde{M}$, $\{\tilde{W}_0\} \cup \{\tilde{W}_\alpha^{(j)}\}_\alpha$, and $\{\tilde{\chi}_0\} \cup \{\tilde{\chi}_\alpha^{(j)}\}_\alpha$, $j = 1, 2$. The $\tilde{W}_\alpha^{(j)}$ are copies of W_α for each side of M and $\tilde{\chi}_\alpha^{(j)}$ is just χ_α on W_α for that side.

If g_1 is integrable on $\tilde{M}$, then

$$\begin{aligned}
&\int_{\tilde{M}} g_1 \, dM \\
&= \int_{\tilde{M}} \tilde{\chi}_0 \cdot g_1 \, dM \ + \ \sum_{\alpha,j} \int_{\tilde{M}} \tilde{\chi}_\alpha^{(j)} \cdot g_1 \, dM \\
&= \int_{\tilde{W}_0} \tilde{\chi}_0 \cdot g_1 \, dM \ + \ \sum_{\alpha,j} \int_{\tilde{W}_\alpha^{(j)}} \tilde{\chi}_\alpha^{(j)} \cdot g_1 \, dM \\
&= \int_{\tilde{W}_0} \tilde{\chi}_0 \cdot g_1 \, dM \ + \ \sum_{\alpha,j} \int_{W_\alpha} \chi_\alpha \cdot g_1 \, dM \qquad (80)
\end{aligned}$$

where on the RHS of the last line, we evaluate the multivalued g_1 on the side corresponding to $W_\alpha^{(j)}$.

We consider one of the integrals, so we assume that $h_1 = \chi_\alpha \cdot g_1$ has compact support in one W_α on only one side of M.

Lemma 7.1 *In the preceding situation*

$$\int_{W_\alpha} h_1 \, dM \ = \ \int_{V_\alpha} \int_{\tilde{M}_x} h_1 \, d\bar{M}_x dV_\Gamma \qquad (81)$$

Proof of Lemma 7.1 : We let $W'_\alpha = V'_\alpha \times U'_\alpha \subset \mathbb{R}^n$ be an open subset and $\phi : W'_\alpha \to W_\alpha$ a smooth parametrization so that $p \circ \phi(x, y) = \phi_0(x)$ for $\phi_0 : V'_\alpha \to V_\alpha$ also a smooth parametrization. Then,

$$\int_{W_\alpha} h_1 \, dM \ = \ \int_{W_\alpha} h_1 \cdot \rho \, dV \ = \ \int_{W'_\alpha} (h_1 \cdot \rho) \circ \phi \ \phi^* dV$$

We compute

$$dV(w_1, \ldots, w_n) \ = \ \det(\mathbf{n}, w_1, \ldots, w_n)$$

for $\mathbf{n}$ normal to M. Then,

$$\phi^* dV(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_k}, \frac{\partial}{\partial u_1}, \ldots, \frac{\partial}{\partial u_{n-k}}) \ = \ dV(\frac{\partial \phi}{\partial x_1}, \ldots, \frac{\partial \phi}{\partial x_k}, \frac{\partial \phi}{\partial u_1}, \ldots, \frac{\partial \phi}{\partial u_{n-k}})$$

From $p \circ \phi(x, y) = \phi_0(x)$, and letting $M_z = p^{-1}(z)$ we have

$$\frac{\partial \phi}{\partial x_i} \ = \ \frac{\partial \phi_0}{\partial x_i} + w_i \ = \ \sum_{j=1}^{k} a_{ij} e_j + w_i$$

for $\{e_1, \ldots, e_k\}$ an orthonormal basis for $T_{\phi_0(x)} V_\alpha$ and $w_i \in T_{\phi(x,y)} M_{\phi_0(x)}$. We may also write

$$\frac{\partial \phi}{\partial u_i} \ = \ \sum_{j=1}^{n-k} b_{ij} e'_j$$

for $\{e'_1, \ldots, e'_{n-k}\}$ an orthonormal basis for $T_{\phi(x,y)} M_{\phi_0(x)}$.

Let $A = (a_{ij})$ and $B = (b_{ij})$, Then, we may expand

$$\begin{aligned}
&dV(\frac{\partial \phi}{\partial x_1}, \ldots, \frac{\partial \phi}{\partial x_k}, \frac{\partial \phi}{\partial u_1}, \ldots, \frac{\partial \phi}{\partial u_{n-k}}) \\
&= \ \det(B) dV(\frac{\partial \phi}{\partial x_1}, \ldots, \frac{\partial \phi}{\partial x_k}, e'_1, \ldots, e'_{n-k}) \\
&= \ \det(B) \cdot \det(A) \cdot dV(e_1, \ldots, e_k, e'_1, \ldots, e'_{n-k}) \\
&= \ \det(B) \cdot \det(A) dV(e''_1, \ldots, e''_k, e'_1, \ldots, e'_{n-k}) \qquad (82)
\end{aligned}$$

where $dp(e''_i) = e_i$. Then, by the definition of ν, the right hand side of (82) equals $\det(B) \cdot \det(A) \cdot \nu(\phi(x, y))$. Thus, the integral becomes

$$\begin{aligned}
&\int_{W'_\alpha} \phi^*(h_1 \cdot \rho \cdot dV) \ = \\
&\quad \int_{W'_\alpha} (h_1 \cdot \rho) \circ \phi \cdot \det(B) \cdot \det(A) \cdot \nu(\phi(x, y)) \\
&\qquad dx_1 \ldots dx_k dy_1 \ldots dy_{n-k} \qquad (83)
\end{aligned}$$

Then, $\det(B) \cdot dy_1 \ldots dy_{n-k}$ is the pull-back of the Riemannian volume dA_t on $M_{\phi_0(x,y)}$, and $\det(A) \cdot$

$dx_1 \dots dx_k$ is the pull-back of dV_{V_α}. By changing the order of integration in (83) we obtain

$$\int\limits_{W'_\alpha} h_1\, dM$$

$$= \int\limits_{V'_\alpha} \left(\int\limits_{U'_\alpha} h_1 \cdot (\rho \cdot \nu) \circ \phi \cdot \det(B) dy_1 \dots dy_{n-k} \right) \cdot \det(A)\, dx_1 \dots dx_k$$

$$= \int\limits_{V_\alpha} \int\limits_{\tilde{M}_x} h_1\, d\bar{M}_x dV_\Gamma \qquad (84)$$

□

By Lemma 7.1 applied to each integral on the RHS of (80), we obtain

$$\int\limits_{\tilde{M}} g_1\, dM$$

$$= \int\limits_{\tilde{M}} \tilde{\chi}_0 \cdot g_1\, dM \;+\; \sum_{\alpha,j} \int\limits_{\Gamma} \int\limits_{\tilde{M}_x} \tilde{\chi}^{(j)}_\alpha \cdot g_1\, d\bar{M}_x dV_\Gamma$$

$$= \int\limits_{\tilde{M}} \tilde{\chi}_0 \cdot g_1\, dM \;+\; \int\limits_{\Gamma} \int\limits_{\tilde{M}_x} (1 - \tilde{\chi}_0) \cdot g_1\, d\bar{M}_x dV_\Gamma \quad (85)$$

Then, since we choose a sequence $\varepsilon \to 0$ and a decreasing sequence of tubular systems N_ε whose intersection is $\Sigma \cup M_{sing}$, so $\chi_0 \to 0$ as $\varepsilon \to 0$. Then, applying the dominated convergence theorem to each term on the RHS, we obtain

$$\int\limits_{\tilde{M}} g_1 dM \;=\; 0 + \int\limits_{\Gamma} \int\limits_{\tilde{M}_x} g_1\, d\bar{M}_x dV_\Gamma \qquad (86)$$

Finally, using (86) with g_1 replaced by $\bar{g}$ defined by (31), we obtain the result. □

Proof of Theorem 4.12 : We follow the same line of argument as for the proof of Theorem 4.5, except that $W'_\alpha = V'_\alpha \times U'_\alpha \times [0,1]$ where V'_α parametrizes an open subset of S^{n-k} and U'_α is an open subset of M, which we may assume is a submanifold of $\mathbb{R}^{n+1}$. We let $\tilde{W}'_\alpha = \pi^{-1}(W'_\alpha) \subset \tilde{M}$. For simplicity, in what follows we drop the subscript α.

The parametrization map for the part of Ω obtained from the radial flow from W' is given by

$$\psi(x, \theta, t) \;=\; x \,+\, t \cdot r(x, \theta) \cdot U_1(x, \theta) \qquad (87)$$

where $\theta = (\theta_1, \cdots, \theta_{n-k})$. Then, for a point $y_0 = \psi(x_0, \theta_0, t_0)$, we let $\mathbf{v} = \{v_1, \cdots, v_k\}$ denote a positively oriented orthonormal basis for $T_{x_0}M$, and $\mathbf{w} = \{w_1, \cdots, w_{n-k}\}$ a positively oriented orthonormal basis for S^{n-k} at θ_0 We also let $\mathbf{w}' = \{w'_1, \cdots, w'_{n-k}\}$ be their images under $d\psi(x_0, \theta, 1)$ (so that $\{U_1, w'_1, \cdots, w'_{n-k}, v_1, \cdots, v_k\}$ is positively priented for $\mathbb{R}^{n+1}$.

Then, we compute

$$\frac{\partial \psi}{\partial t} \;=\; rU_1; \quad \frac{\partial \psi}{\partial w_j} = t \frac{\partial r}{\partial w_j} U_1 \,+\, t \cdot r \frac{\partial U_1}{\partial w_j} \qquad (88)$$

and

$$\frac{\partial \psi}{\partial v_i} \;=\; v_i \,+\, t \frac{\partial r}{\partial v_i} U_1 \,+\, t \cdot r \frac{\partial U_1}{\partial v_i} \qquad (89)$$

We also have,

$$\frac{\partial U_1}{\partial w_j} \;=\; w'_j \quad \text{and} \quad \frac{\partial U_1}{\partial v_i} \;=\; z_j - S_{rel}(v_i) \qquad (90)$$

with $z_j \in \Pi_{t_0}$.

Then, by using (88), (89) and (90), we may compute

$$\psi^* dV(\frac{\partial}{\partial t}, w_1, \cdots, w_{n-k}, v_1, \cdots, v_k) \;=\; \det(d\psi(t), d\psi(w_1), \dots, d\psi(v_k))$$

which equals

$$\begin{aligned}
&= r \det(U_1, d\psi(w_1), \dots, d\psi(v_k)) \\
&= t^{n-k} r^{n-k+1} \det(U_1, w'_1, \dots, w'_{n-k}, d\psi(v_1), \dots, d\psi(v_k)) \\
&= t^{n-k} r^{n-k+1} \det(U_1, w'_1, \dots, w'_{n-k}, (I - trS_{rel})(v_1), \dots, (I - tr \cdot S_{rel})(v_k)) \\
&= t^{n-k} r^{n-k+1} \det(I - tr \cdot S_{rel}) \cdot \det(U_1, w'_1, \dots, w'_{n-k}, v_1, \dots, v_k) \\
&= t^{n-k} r^{n-k+1} \det(I - tr \cdot S_{rel}) \nu(y_0) \qquad (91)
\end{aligned}$$

Hence,

$$\psi^* dV \;=\; t^{n-k} r^{n-k+1} \det(I - tr \cdot S_{rel}) \cdot \nu \cdot dt\, dS\, dA \qquad (92)$$

for dS the volume form on S^{n-k} and dA is the volume form on M.

We again use the change of variables formula.

$$\int\limits_{W'_\alpha} \chi_\alpha \cdot g_1\, dV$$

$$= \int\limits_{V'_\alpha} \int\limits_{U'_\alpha} \int\limits_0^1 \chi_\alpha \cdot g_1 \cdot t^{n-k} r^{n-k+1} \det(I - tr \cdot S_{rel}) \cdot$$

$$\nu dt\, dS\, dA$$

$$= \int_{V'_\alpha}\int_{U'_\alpha}(\int_0^1 \chi_\alpha \cdot g_1 \cdot t^{n-k} \cdot \det(I - tr \cdot S_{rel})dt) \cdot r^{n-k+1}\, dS\, d\bar{M}$$

$$= \int_{M}\int_{S^{n-k}}(\int_0^1 \chi_\alpha \cdot g_1 \cdot t^{n-k} \cdot \det(I - tr \cdot S_{rel})dt) \cdot r^{n-k+1}\, dS d\bar{M} \quad (93)$$

Summing (93) over α yields the desired result. □

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