SEMI-COHERENCE FOR SEMIANALYTIC SETS AND STRATIFICATIONS AND SINGULARITY THEORY OF MAPPINGS ON STRATIFICATIONS

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To David Trotman on his Sixtieth Birthday
With wishes of many more Happy and Successful Years in Mathematics

Abstract. We consider the conditions on a local stratification $\mathcal{V}$ which ensure that the local singularity theory in the sense of Thom-Mather, such as finite determinacy, versal unfolding, and classification theorems and their topological versions apply either to mappings on the stratified set $\mathcal{V}$ or for an equivalence of mappings which preserve $\mathcal{V}$ in source or target for any of the categories: complex analytic, real analytic, or smooth. For such a stratification $\mathcal{V}$, it is sufficient that the equivalence group be a "geometric subgroup of $\mathcal{A}$ or $\mathcal{K}$", and this reduces to the structure of the module $\mathrm{Derlog}(\mathcal{V})$ of germs of vector fields on the ambient space which are tangent to $\mathcal{V}$. In the holomorphic or real analytic categories, with holomorphic, resp. real analytic stratifications, we show the necessary conditions are satisfied.

However, in the smooth category the general question is open for smooth stratifications. We introduce a restricted class of "semi-coherent"semianalytic stratifications $(\mathcal{V}, 0)$ and semianalytic set germs $(V, 0)$ (and their diffeomorphic images). This notion generalizes Malgrange's notion of "real coherence"for real analytic sets. It is defined in terms of both $\mathrm{Derlog}(\mathcal{V})$ and $I(V)$ (the ideal of smooth function germs vanishing on $(V, 0)$) being finitely generated modulo infinitely flat vector fields, resp. functions. This class includes the *special semianalytic stratifications and sets* in [DGH], and semianalytic sets such as Maxwell sets, "medial axes/central sets", and the discriminants of C^∞-stable germs in the nice dimensions. We further show that the equivalence groups in the smooth category for these stratifications are then geometric subgroups of $\mathcal{A}$ or $\mathcal{K}$.

Introduction

For a stratification $\mathcal{V}$ of a germ $(V, 0)$, we consider singularity theory in the Thom-Mather sense for mappings $f : \mathbf{k}^n, 0 \to \mathbf{k}^p, 0$ either on $\mathcal{V}$ or by an equivalence preserving $\mathcal{V}$. in any of the categories: holomorphic (with $\mathbf{k} = \mathbb{C}$), real analytic, or smooth (for $\mathbf{k} = \mathbb{R}$). Traditionally, the main interests in stratifications $\mathcal{V}$ has involved their properties and the consequences for equisingularity of varieties and mappings as a result of the work of many people beginning with Whitney[Wh], Thom [Th], Hironaka [H1, H2] Lojasiewicz [Lo], Mather [M1] and further built

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upon by David Trotman with his many coworkers and students, e.g. [Tr1, Tr2, BTr, NTr, OTr, MPT, TrW], along with the important contributions by Verdier [Ve], Mostowski [Ms], Hardt [Ht], and many others. By contrast, singularity theory on a given stratified variety V has concentrated on the topological properties of V, either computed via stratified Morse functions on V, using Stratified Morse Theory of Goresky-MacPherson [GM] or generic projections of Lê and Teissier [LeT].

For mappings on varieties $(V, 0)$ or equivalences preserving varieties, singularity-theoretic results have concerned: infinitesimal stability implies stability for a holomorphic germs on holomorphic $(V, 0)$, Galligo [Ga]; finite determinacy modulo an ideal (= I(V)), DuPlessis-Gaffney [DPG]; and the classification of function germs under $\mathcal{R}$-equivalence preserving a hypersurfaces $(V, 0)$ in several specific cases, Arnold [A] and Lyashko [Ly]. Also, a classification of low dimensional smooth germs has been carried out with $(V, 0)$ denoting either a smooth curve on a surface (or surface with boundary) Bruce-Giblin [BG] and Goryunov [Go], or "creases and corners" Tari [Ta1, Ta2].

These latter results fit into the general framework where for any of the three categories, a group of germs of diffeomorphisms of $(\mathbf{k}^n, 0)$, denoted by $\mathcal{D}_n$, is replaced by a group $\mathcal{D}_V$ which preserves a subspace $V, 0 \subset \mathbf{k}^n, 0$. In the holomorphic or real analytic categories, $(V, 0)$ can be the germ of any holomorphic, resp. real analytic set germ. However, in the smooth category, the results have been limited to $(V, 0)$ which are smooth diffeomorphic images of *real coherent analytic germs* in the sense of Malgrange [Mg]. Then, for example, for any of the standard equivalences in the Thom-Mather sense, $\mathcal{G} = \mathcal{R}$, $\mathcal{K}$, or $\mathcal{A}$, we may replace the group of diffeomorphisms in the source or target by the appropriate $\mathcal{D}_V$, and obtain the corresponding group $\mathcal{G}_V$ preserving $V, 0$ in the target, or ${}_V\mathcal{G}$ preserving $V, 0$ in the source. Second, we may further enlarge the equivalence group to yield equivalences $\mathcal{G}(V)$ capturing equivalence of germs on $V, 0$, and even allow both the variety $V, 0$ to vary along with the mappings.

The basic theorems of singularity theory are valid for these equivalences, because each of the groups $\mathcal{G}_V$, ${}_V\mathcal{G}$, or $\mathcal{G}_V$ are "geometric subgroup of $\mathcal{A}$ or $\mathcal{K}$"(with an adequately ordered system of algebras) in the sense of Damon [D2]. All of the four conditions to be such a group are naturally satisfied except for the tangent space condition which requires that the tangent space $T\mathcal{G}_e$ be finitely generated as a module over the system of algebras (and in the smooth case this can be relaxed to hold modulo infinitely flat vector fields, see [D1] and [D3, §8]). In the holomorphic or real analytic categories, the tangent space $\mathrm{Derlog}(V) = T\mathcal{D}_{V,e}$ (see §1) is finitely generated over the appropriate ring of germs, and in the smooth category for real coherent analytic germs $(V, 0)$, this is true (modulo infinitely flat vector fields, by [D1, Lemma 1.1]). As a consequence, the basic theorems of singularity theory are valid for these equivalences including: the finite determinacy theorem, versal unfolding theorem, and infinitesimal stability implies stability under deformations, and classification theorems.

Here we address two questions. First, in a number of situations of interest we wish to replace $(V, 0)$ by a stratification $(\mathcal{V}, 0)$ of a set germ $(V, 0)$ in the appropriate category; and furthermore, in the smooth category we would additionally like to allow the stratification $(\mathcal{V}, 0)$ and the set germ $(V, 0)$ to be semianalytic. Several examples where these conditions play a role involve: discriminants of stable germs, which in general are only (diffeomorphic to) semialgebraic sets; the Blum medial axis (or central set) for generic smooth regions in $\mathbb{R}^n$ are locally diffeomorphic to semialgebraic sets, and in computer vision, the stratifications which are needed to describe the geometric features of natural objects, and the refinements of these stratifications resulting from shade and shadows requires the consideration of semianalytic stratifications.

The first goal is to extend Malgrange's notion of real coherence for real analytic germs to a sufficently large class of semianalytic sets and stratifications. In the smooth category, A real coherent germ $(V, 0)$ in the sense of Malgrange has the property that the ideal $I(V)$ of smooth germs vanishing on $(V, 0)$ is finitely generated over the ring of smooth germs $\mathcal{E}_n$ by the generators of $I(V)^{an}$, the ideal of real analytic germs vanishing on $(V, 0)$ (see [Mg, Chap. VI, Theorem 3.10]). However, to be applicable to the equivalence groups described above, it was also necessary to have that the module $\mathrm{Derlog}(V)$ is finitely generated (modulo infinitely flat vector fields in the smooth category). We ask if there is a generalization of Malgrange's notion of being real coherent which will apply to these semianalytic sets and stratifications? Secondly, is this generalization useful to establish that the corresponding equivalence groups are geometric subgroups of $\mathcal{A}$ or $\mathcal{K}$?

We shall give a positive answer to both of these questions. We introduce a notion of *semi-coherence for semianalytic sets and stratifications*, which concerns the finite generation of both the ideal $I(V)$ and $\mathrm{Derlog}(V)$ (or the corresponding ideals and modules for a stratification $\mathcal{V}$) modulo infinitely flat vector fields. Besides having several naturality properties, this notion includes the three classes of semianalytic sets and stratifications described above, including the class of *special semianalytic sets and stratifications* introduced in [DGH]; and it establishes that the corresponding equivalence groups are geometric subgroups of $\mathcal{A}$ or $\mathcal{K}$ so that the basic theorems of singularity theory are valid for smooth mappings under such an equivalence preserving the stratification or for germs on the stratification. These results are used in [DGH] for the classification of local features of images of objects with geometric features inlcuding shade and shadows.

In §1 we recall Malgrange's notion of being real coherent and give several examples due to Malgrange and Whitney of analytic sets which do not satisfy the condition. Next, we introduce the more general notion of semi-coherence for semianalytic sets and explain how this condition includes the class of *special semianalytic sets* introduced in [DGH]. We also prove that the class of weighted homogeneous semianalytic germs are semi-coherent. This includes examples of analytic sets that are not real coherent and in addition the discriminants of stable germs in the nice dimensions. In §2, we extend the notion of semi-coherence to semianalytic stratifications and give several conditions that insure that a semianalytic stratification is semi-coherent, including the class of *special semianalytic stratifications* in [DGH]. In §3, we briefly indicate how the the resulting equivalence groups satisfy the conditions for being geometric subgroups. In §4, we give the proofs of several of the results and indicated how the others follow by slightly modifying the proofs in [DGH] for the special semianalytic stratifications.

1. Semi-coherent Semianalytic Sets

In this section we consider the smooth category, except we consider a semianalytic set $V, 0 \subset \mathbb{R}^n, 0$ with local analytic Zariski closure $(\tilde{V}, 0)$. We will simultaneously consider both the rings of smooth germs $\mathcal{E}_n$ with maximal ideal denoted by

m_n, and real analytic germs $\mathcal{A}_n$. We let θ_n denote the module of germs of smooth vector fields on $(\mathbb{R}^n, 0)$. Then, we let $I(V)$ denote the ideal of smooth germs $f \in \mathcal{E}_n$ which vanish on V in a neighborhood of 0, and $I^{an}(V)$ the corresponding ideal of analytic germs. In general, it is not known when $I(V)$ is a finitely generated ideal in $\mathcal{E}_n$. Malgrange [Mg] introduced the notion of V being real coherent, which means that there is a set of generators $\{g_1, g_2, \dots, g_k\}$ for $I^{an}(V)$ and a neighborhood U of 0 on which they are defined so that for $x \in U$, the germs of the g_i at x generate the ideal of real analytic germs at x vanishing on (V, x). He then proves that for such a real coherent analytic germ $(V, 0)$, $I(V) = I^{an}(V) \cdot \mathcal{E}_n$, so in particular it is finite generated [Mg].

We let $\mathrm{Derlog}^{an}(\tilde{V})$ denote the module of real analytic vector fields ξ satisfying $\xi(I^{an}(\tilde{V})) \subset I^{an}(\tilde{V})$. It is a finitely generated $\mathcal{A}_n$-module. We let $\mathcal{V}$ denote the canonical Whitney stratification of $(V, 0)$. Then, we define

$$\mathrm{Derlog}(V) \;=\; \{\xi \in \theta_n : \xi \text{ is tangent to the strata of } \mathcal{V}\} \tag{1.1}$$

Remark 1.1. If $\xi \in \mathrm{Derlog}(V)$ and $g \in I(V)$, then as g vanishes on the strata of $\mathcal{V}$, $\xi(g)$ vanishes on the strata of $\mathcal{V}$, and hence on $(V, 0)$, so $\xi(g) \in I(V)$.

Moreover, if ξ is analytic and $g \in I^{an}(\tilde{V})$, then again g vanishes on the strata of $\mathcal{V}$, so $\xi(g)$ vanishes on $(V, 0)$ and hence on its local analytic Zariski closure $\tilde{V}$ so $\xi(g) \in I^{an}(\tilde{V})$ and $\xi \in \mathrm{Derlog}^{an}(\tilde{V})$.

Also, if $(V, 0)$ is real coherent in the sense of Malgrange, then by an argument in [D1, §1], if $\xi(I(V)) \subset I(V)$, then $\xi \in \mathrm{Derlog}(V)$ as defined in (1.1). Thus, $\mathrm{Derlog}(V)$ may be alternately be defined by the condition $\xi(I(V)) \subset I(V)$ as in [D1, §1], except there the notation θ_V was used.

The notation $\mathrm{Derlog}(V)$ is a variant of the notation introduced by Saito [Sa] for the module of "logarithmic vector fields" for a complex hypersurface singularity $V, 0$, reflecting the relation with logarithmic forms.

However, even for real coherent analytic germs it is generally unknown whether $\mathrm{Derlog}(V)$ is a finitely generated $\mathcal{E}_n$ module. A weaker result which is satisfactory for many applications in singularity theory is the following (see [D1, Lemma1.1]).

Proposition 1.2. *If $V, 0 \subset \mathbb{R}^n, 0$ is real coherent then*

$$\mathrm{Derlog}(V) \;\equiv \mathcal{E}^n\{\zeta_1, \dots, \zeta_r\} \quad \mathrm{mod}\ m_n^\infty \theta_n$$

where $\{\zeta_1, \dots, \zeta_r\}$ are a set of generators of $\mathrm{Derlog}^{an}(V)$.

Here m_n^∞ denotes the ideal of infinitely flat function germs.

By the result in [D3, §8], in the smooth category, for a real coherent analytic germ $V, 0 \subset \mathbb{R}^n$, we may replace $\mathcal{D}_n$ by $\mathcal{D}_V$ in any standard group of equivalences $\mathcal{G}$ and conclude they are geometric subgroups of $\mathcal{A}$ or $\mathcal{K}$. However, this places an excessive restriction even for real analytic $(V, 0)$, and does not address the case of semianalytic $V, 0$. We illustrate the issue with several examples due to Malgrange and Whitney.

Example 1.3 (Malgrange Umbrellas)**.** The following examples are generalizations of that given by Malgrange in [Mg, Example after Def. 3.9, Chap. VI]. We consider $V, 0 \subset \mathbb{R}^{n+1}, 0$ defined by

$$x_{n+1} \cdot \left(\sum_{i=1}^{n} x_i^2\right) \;=\; f(x_1, \dots, x_n)\,,$$

where f is homogeneous of degree $k \geq 3$. Then, the x_{n+1}-axis lies in V and is an isolated line, for if we consider any line $x_i = tb_i$ for $i = 1, \dots n$, with some $b_i \neq 0$, then

$$x_{n+1} \;=\; t^{k_2} \cdot \left(\frac{f(b_1, \dots, b_n)}{\sum_{i=1}^{n} b_i^2} \right)$$

Also, $(V, 0)$ is not real coherent as at a point $x' = (0, \dots, 0, x_{0,n+1})$ with $x_{0,n+1} \neq 0$, (V, x') is locally defined by $x_1 = \cdots = x_n = 0$, and is not generated by the single generator

$$G \;=\; x_{n+1} \cdot \left(\sum_{i=1}^{n} x_i^2 \right) \;-\; f(x_1, \dots, x_n).$$

If $f(x_1, \dots, x_n) > 0$ when some $x_i \neq 0$, then we can remove the handle on the negative x_{n+1}-axis by adding the condition $x_{n+1} \geq 0$ and obtaining a germ of a semianalytic set whose Zariski closure is $(V, 0)$.

Example 1.4 (Generalized Whitney Umbrellas)**.** The standard Whitney umbrella is the image $V = D(F)$ of the stable map germ $F : \mathbb{R}^2, 0 \to \mathbb{R}^3, 0$, where $(y_1, y_2, y_3) = F(x_1, x_2) = (x_1, x_1 x_2, x_2^2)$. It is semialgebraic with analytic Zariski closure $\tilde{V}, 0$ defined by $y_2^2 = y_3 y_1^2$. It has a handle consisting of the y_3 axis with $y_3 > 0$. As for the Malgrange umbrellas, $(\tilde{V}, 0)$ is not real coherent.

More generally we can define "generalized Whitney umbrellas"as images of maps $F : \mathbb{R}^{n+1}, 0 \to \mathbb{R}^{n+2}, 0$ given by

$$(y_1, \dots, y_{n+2}) \;=\; F(x_1, \dots, x_{n+1}) \;=\; (x_1, \dots, x_n, x_{n+1} \cdot f(x_1, \dots, x_n, x_{n+1}^2), x_{n+1}^2)$$

where both f and $f(x_1, \dots, x_n, 0)$ have isolated singularities. Such F are finitely $\mathcal{A}$-determined (see Mond [Mo] for the case $n = 1$); and such images are semialgebraic with Zariski closure $\tilde{V}$ defined by $G = y_{n+1}^2 - y_{n+2} f(y_1, \dots, y_n, y_{n+1}) = 0$.

If $f(x_1, \dots, x_n, x_{n+1}^2)$ is weighted homogeneous of weight c for positive weights $\mathrm{wt}\,(x_i) = b_i > 0$, then both F and G are weighted homogeneous (with $\mathrm{wt}\,(y_i) = b_i$ for $i \leq n$, $\mathrm{wt}\,(y_{n+1}) = b_{n+1} + c$ and $\mathrm{wt}\,(y_{n+2}) = b_{n+2}$ satisfying $b_{n+2} = 2b_{n+1} + c$. In the case that $f(x_1, \dots, x_n, 0) > 0$ whenever some $x_i \neq 0$, then $\tilde{V}$ has a handle consisting of the negative y_{n+2}-axis. Again, it is not real coherent.

Next, we consider more generally $V, 0 \subset \mathbb{R}^n, 0$ a closed semianalytic set in the smooth category. We introduce a notion of $(V, 0)$ being *semi-coherent* which extends that of real coherence of Malgrange to closed semianalytic sets in a form which makes it sufficient for many applications in singularity theory. For $V, 0 \subset \mathbb{R}^n, 0$ which is closed and semianalytic, we let $(\tilde{V}, 0)$ denote its local analytic Zariski closure. We also define $\mathrm{Derlog}(V)$ for a semianalytic set $(V, 0)$ with canonical Whitney stratification $\mathcal{V}$, by (1.1). Then, we define

Definition 1.5. A closed semianalytic set germ $V, 0 \subset \mathbb{R}^n, 0$ will be said to be *semi-coherent* in the smooth category if the following two conditions are satisfied.

i) $I(V) \equiv \mathcal{E}_n\{g_1, \dots g_s\} \mod m_n^{\infty}$,
where $\{g_1, \dots g_s\}$ generate $I^{an}(\tilde{V})$; and

ii) $\mathrm{Derlog}(V) \equiv \mathcal{E}_n\{\zeta_1, \dots, \zeta_r\} \mod m_n^{\infty}\theta_n$
where $\{\zeta_1, \dots, \zeta_r\}$ are a set of germs in $\mathrm{Derlog}^{an}(\tilde{V})$ which are tangent to the strata of $\mathcal{V}$.

Here m_n^∞ denotes the ideal of infinitely flat smooth germs.

More generally a germ $V, 0 \subset \mathbb{R}^n, 0$ is *semi-coherent* if there is a germ of a smooth diffeomorphism $\varphi : \mathbb{R}^n, 0 \to \mathbb{R}^n, 0$ and a semi-coherent semianalytic set $V', 0 \subset \mathbb{R}^n, 0$ such that $\varphi(V') = V$. We shall refer to the semi-coherent semianalytic set $(V', 0)$ as the *semianalytic model* for $(V, 0)$.

It follows by the same argument in [D3, §8], that $V, 0 \subset \mathbb{R}^n, 0$ being semi-coherent is sufficient to be able to conclude the unfolding and determinacy theorems and their consequences are valid for the equivalence groups in the smooth category preserve $(V, 0)$ or for equivalences of smooth germs on $(V, 0)$ (see also §3).

By the result of Malgrange and Proposition 1.2, real coherent analytic germs $(V, 0)$ are semi-coherent. A recent result Damon-Giblin-Haslinger [DGH] identifies a class of *special semianalytic germs* which are semi-coherent. A semianalytic set germ $V, 0 \subset \mathbb{R}^n, 0$ is a special semianalytic germ if its Zariski analytic closure $\tilde{V}, 0$ is real coherent and it satisfies conditions i) and ii) in definition 1.5. This allowed several important classes of semianalytic set germs which are semi-coherent to be identified using a *special semianalytic criteria* to be described in §2. However, for example, the discriminants of stable map germs and the classes of Malgrange and Whitney and umbrellas cannot satisfy the criterion for being special semianalytic set germs as their Zariski closures are not in general real coherent. This leads to the question.

Basic Question: When are semianalytic sets semi-coherent?

We give two distinct types of criteria for a semianalytic set to be semicoherent. The first simple criterion is given by the following.

Proposition 1.6. *Let $V, 0 \subset \mathbb{R}^n, 0$ be semianalytic with local analytic Zariski closure $\tilde{V}, 0 \subset \mathbb{R}^n, 0$. Suppose that $\tilde{V}, 0$ is weighted homogeneous (for positive weights) and that V is invariant under the corresponding $\mathbb{R}_+$-action. Then, $V, 0$ is semi-coherent.*

A consequence of Proposition 1.6 is that both the weighted homogeneous analytic and semianalytic Malgrange and Whitney umbrellas are semi-coherent, even though the analytic versions are not in general real coherent. Thus, the notion of semi-coherence is a more general notion than real coherence for analytic set germs $(V, 0)$. There follows a basic consequence for discriminants of C^∞ stable germs.

Theorem 1.7. *Let $f : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$ be a simple C^∞ stable germ, which includes those in the nice range of dimensions. Then the discriminant $(D(f), 0)$ is semi-coherent.*

Proof of the Theorem. By Mather's classfication theorems for such simple stable germs (see [MIV], and [MVI]), f is $\mathcal{A}$-equivalent to a polynomial germ $g : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$ which is weighted homogeneous of positive weights. Thus, there are germs of diffeomorphisms $\psi : \mathbb{R}^n, 0 \to \mathbb{R}^n, 0$ and $\varphi : \mathbb{R}^p, 0 \to \mathbb{R}^p, 0$ so that $f = \varphi \circ g \circ \psi$. Hence, $\varphi(D(g)) = D(f)$, and it is sufficient to show that $(D(g), 0)$ is semi-coherent. However, as g is a polynomial mapping, it follows by the Tarski-Seidenberg theorem that the image $D(g) = g(\Sigma(g))$ of the singular set $\Sigma(g)$ is semialgebraic, so in particular, semianalytic.

Also, as g is weighted homogeneous for positive weights, so is the Zariski closure $\tilde{D}(g)$ (the complexification $g_{\mathbb{C}}$ has discriminant $D(g_{\mathbb{C}})$ which is weighted homogeneous for positive weights, and $D(g_{\mathbb{C}}) \cap \mathbb{R}^p$ is the Zariski closure of $D(g)$). Furthermore, if $y_0 = g(x_0) \in D(g)$ with $x_0 \in \Sigma(g)$, then by the weighted homogeneity of g, $\mathbb{R}_+ \cdot x_0 \subset \Sigma(g)$ and $g(\mathbb{R}_+ \cdot x_0) = \mathbb{R}_+ \cdot y_0$, so $\mathbb{R}_+ \cdot y_0 \subset D(g)$. Thus, by Proposition 1.6, $(D(g), 0)$, and hence $(D(f), 0)$, are semi-coherent. □

Next, we illustrate that even for the simplest semianalytic germs that the equalities in Definition 1.5 are only true modulo infinitely flat functions and vector fields.

Example 1.8. Let $V, 0 \subset \mathbb{R}^n, 0$ denote the model for a k-corner. It is defined by $f = 0$ where $f(x_1, \dots, x_k) = \prod_{i=1}^k x_i$ and the inequalities $x_i \geq 0$ for $i = 1, \dots, k$. Its local analytic Zariski closure $\tilde{V}, 0$ is the germ defined by $f = 0$. The module $\mathrm{Derlog}^{an}(V)$ of germs of analytic vector fields tangent to V is generated by $x_i \dfrac{\partial}{\partial x_i}$, $i = 1, \dots, k$ and $\dfrac{\partial}{\partial x_j}$, $j = k+1, \dots, n$. We exhibit an infinitely flat smooth germ $g \in I(V)$, but not in the ideal $(f) \cdot \mathcal{E}_n$, and infinitely flat smooth germs of vector fields $g \dfrac{\partial}{\partial x_i} \in \mathrm{Derlog}(V)$, $i = 1, \dots, k$, which are not in $\mathcal{E}_n\{x_i \dfrac{\partial}{\partial x_i}, i = 1, \dots, k; \dfrac{\partial}{\partial x_j}, j = k+1, \dots, n\}$.

Let $\rho(x)$ be the infinitely flat germ

$$\rho(x) \;=\; \begin{cases} \exp(-\frac{1}{x^2}) & x < 0, \\ 0 & x \geq 0 \end{cases} .$$

Let $g(x_1, \dots, x_n) = \sum_{i=1}^k \rho(x_i)^2$. Then, g vanishes on V. We claim it is not smoothly divisible by x_i for any $i = 1, \dots, k$. For example, if g were smoothly divisible by x_1, then as $\rho(x_1)$ is smoothly divisible by x_1, so would be $g - \rho(x_1)^2 = \sum_{i=2}^k \rho(x_i)^2$. However, $\sum_{i=2}^k \rho(x_i)^2$ is not smoothly divisible by x_1. A similar argument works for not being smoothly divisible x_i for $i = 2, \dots, k$. Thus, $g \notin (f) \cdot \mathcal{E}_n$. Also, if $g \dfrac{\partial}{\partial x_1} \in \mathcal{E}_n\{x_i \dfrac{\partial}{\partial x_i}, i = 1, \dots, k; \dfrac{\partial}{\partial x_j}, j = k+1, \dots, n\}$, then $g \dfrac{\partial}{\partial x_1} = h \cdot x_1 \dfrac{\partial}{\partial x_1}$. This would imply x_1 smoothly divides g, which, as we just saw, is impossible. There is an analogous argument for $i = 2, \dots, k$.

We note that we could replace ρ by any infinitely flat function which vanishes for $x \geq 0$ but not identically on $\mathbb{R}$. Also, an analogous argument would work for more general semianalytic sets involving more than one inequality.

There is a second criterion, the *special semianalytic criterion* given in [DGH], which applies to semianalytic sets that are not necessarily weighted homogeneous and will yield *special semianalytic stratifications*. We describe it in §2.

There are also further properties of both semicoherent semianalytic sets and the special semianalytic sets. However, these properties are best described for the more general notion of semi-coherent semianalytic stratifications to be introduced next.

2. Semi-coherent Semianalytic Stratifications

Let $V, 0 \subset \mathbb{R}^n, 0$ be a germ of a closed semianalytic set, and let $\tilde{V}, 0 \subset \mathbb{R}^n, 0$ be its real local analytic Zariski closure with $I^{an}(V) = I^{an}(\tilde{V})$ the ideal of real analytic germs vanishing on $(V, 0)$ and defining $\tilde{V}$. By a *semianalytic stratification* $\mathcal{V}$ of $(V, 0)$ we mean a decreasing sequence of closed semianalytic set germs $V = V_k \supset V_{k-1} \supset \cdots \supset V_1 \supset V_0 = \{0\}$ with $\dim V_j = j$ and $V_j \backslash V_{j-1}$ consisting of strata of dimension j. For the stratification $\mathcal{V}$, we define for the smooth category

$$\text{Derlog}(\mathcal{V}) = \{\xi \in \theta_n : \xi \text{ is tangent to the strata } S_i \text{ of } \mathcal{V} \text{ for all } i\}. \tag{2.1}$$

We also consider $\text{Derlog}^{an}(\tilde{V})$ in the real analytic category. Then, we define

Definition 2.1. The stratification $\mathcal{V}$ of the germ of the closed semianalytic set $V, 0 \subset \mathbb{R}^n, 0$ is a *semi-coherent stratification* if it satisfies the following two conditions:

i) if $\{g_1, \dots, g_k\}$ generate $I^{an}(\tilde{V})$, then in the smooth category

$$I(V) \;\equiv\; \mathcal{E}_n\{g_1, \dots, g_k\} \quad \text{mod} \;\; m_n^{\infty};$$

and

ii) there are $\xi_j \in \text{Derlog}^{an}(\tilde{V})$, $j = 1, \dots, m$ which are tangent to the strata S_i of $\mathcal{V}$ for all i such that

$$\text{Derlog}(\mathcal{V}) \;\equiv\; \mathcal{E}_n\{\xi_1, \dots, \xi_m\} \quad \text{mod} \;\; m_n^{\infty} \cdot \theta_n.$$

In general we say that a *stratification* $\mathcal{V}$ *of a germ* $V, 0 \subset \mathbb{R}^n, 0$ *is semi-coherent* if there is a germ of a diffeomorphism $\varphi : \mathbb{R}^n, 0 \to \mathbb{R}^n, 0$ and a semi-coherent stratification $\mathcal{V}'$ of a semianalytic germ $(V', 0)$ such that $\varphi(V') = V$ and $\varphi(\mathcal{V}') = \mathcal{V}$.

If in Definition 2.1, we require the stronger condition that $\tilde{V}$ is real coherent, then the stratification is a *special semianalytic stratification* (SSA stratification) in the sense of [DGH].

Remark 2.2. If $(V, 0)$ is a semi-coherent semianalytic set, then the canonical Whitney stratification $\mathcal{V}$ of $(V, 0)$ is a semi-coherent semianalytic stratification in the sense of Definition 2.1. This follows since vector fields tangent to V are tangent to the canonical Whitney stratification of $(V, 0)$; and conversely by Remark 1.1, any analytic vector field ξ tangent to the Whitney stratification of $(V, 0)$, will satisfy $\xi(g) \in I^{an}(\tilde{V})$ for any $g \in I^{an}(\tilde{V})$. Hence, by property ii) for semi-coherent semianalytic sets, we have

$$\text{Derlog}(\mathcal{V}) \;=\; \text{Derlog}(V) \;\equiv\; \mathcal{E}_n\{\zeta_1, \dots, \zeta_r\} \quad \text{mod} \; m_n^{\infty} \cdot \theta_n \,.$$

Hence, properties for semi-coherent stratifications will hold for semi-coherent semianalytic sets.

The definition of semi-coherent stratification depends upon an ambient space. We first note that the class of semi-coherent stratifications is preserved under two standard operations, which removes this restriction.

Proposition 2.3. *Let $\mathcal{V}$ be a semi-coherent stratification of a semianalytic set germ $V, 0 \subset \mathbb{R}^n, 0$.*

(1) *If $\varphi : \mathbb{R}^n, 0 \to M, p$ is an analytic diffeomorphism to an analytic submanifold $M, p \subseteq \mathbb{R}^m, p$, then the stratification $\varphi(\mathcal{V})$ of $(\varphi(V), p)$ is a semi-coherent stratification.*

(2) *Define a stratification $\mathcal{V}'$ of $V \times \mathbb{R}^k, 0 \subset \mathbb{R}^{n+k}, 0$ which has strata $S'_i = S_i \times \mathbb{R}^k$ for the strata S_i of $\mathcal{V}$. Then $\mathcal{V}'$ is a semi-coherent stratification of $V \times \mathbb{R}^k, 0 \subset \mathbb{R}^{n+k}, 0$.*

The proof of this proposition closely follows the proof of the corresponding result for special semianalytic stratifications [DGH, Prop. 5.4, Chap. 5]; see §4.

Second, we may refine a semi-coherent stratification by a series of semi-coherent stratifications in the following way. Let $\mathcal{V}_i$ be semi-coherent stratifications of closed semianalytic germs $V_i, 0$, $i = 1, \dots, k$, with $V_1, 0 \subset V_2, 0 \subset \dots V_k, 0 \subset \mathbb{R}^n, 0$ such that each stratum of $\mathcal{V}_i$ is contained in a stratum of $\mathcal{V}_{i+1}$ for each $i < k$. Then, we can define a stratification $\mathcal{V}$ of $(V, 0) = (V_k, 0)$ which is a refinement $\mathcal{V}_k$ with strata consisting of $S_i \backslash V_j$ for all S_i in $\mathcal{V}_{j+1}$ and all $1 \leq j < k$, together with the strata of $\mathcal{V}_1$.

Proposition 2.4. *In the preceding situation, the stratification $\mathcal{V}$ of the closed semianalytic germ $V, 0 \subset \mathbb{R}^n, 0$ is a semi-coherent semianalytic stratification.*

To accompany these results, we next give the second criterion for establishing semi-coherence of a stratification $\mathcal{V}$ of a germ of a closed semianalytic set $(V, 0)$, with Zariski closure $(\tilde{V}, 0)$. This is given by the following criterion from [DGH, Def 5.1, Chap 5].

Special Semianalytic Criterion:

Definition 2.5. A stratification $\mathcal{V}$ of $V, 0$ is said to satisfy the *special semianalytic criterion* (SSC) if $\tilde{V}$ is real coherent and the stratification satisfies the following conditions:

(1) V and each of the irreducible components V_i are unions of connected components of the canonical Whitney stratification of $\tilde{V}$.
(2) Each irreducible component $\tilde{V}_i$ of $\tilde{V}$ is smooth; and
(3) For each i, the set of tangent lines $T_0\gamma$ to analytic curves γ in V_i with $\gamma(t) \in V_i$ for $t \geq 0$ and $\gamma(0) = 0$ form a Zariski dense subset of $\mathbb{P}T_0\tilde{V}_i$.

Then, the second criterion is the following given in [DGH, Prop. 5.3, Chap 5].

Proposition 2.6. *A stratification $\mathcal{V}$ of the closed semianalytic germ $V, 0 \subset \mathbb{R}^n, 0$ which satisfies the special semianalytic criterion is a special semianalytic stratification. Moreover,*

$$\mathrm{Derlog}(\mathcal{V}) \;\equiv\; \mathrm{Derlog}(\tilde{V}) \quad \mathrm{mod}\ m_n^{\infty}\, \theta_n \tag{2.2}$$

In order to apply this result we use a simple criterion for an analytic set germ $(V, 0)$ being real coherent. This is given by the following (see [DGH, Chap. 5, Prop. 4.1]).

Proposition 2.7. *Let $V, 0 \subset \mathbb{R}^n, 0$ be a real analytic germ with complexification $V_{\mathbb{C}}, 0 \subset \mathbb{C}^n, 0$. Suppose that there is a neighborhood U of $0 \in \mathbb{R}^n$ such that for $x \in U$, the germ (V, x) is Zariski dense in $(V_{\mathbb{C}}, x)$ for the local analytic Zariski topology at x. Then, V is real coherent.*

We illustrate using these criterion for several examples that occur for natural images where stratifications defining generic geometric features of objects are refined by the stratification resulting from shade/shadow curves from a light source (see [DGH, Chap. 6, 7, 8]). The generic geometric features of objects are modeled by semianalytic sets which are "partial hyperplane arrangements".

Example 2.8 (Partial Hyperplane Arrangements)**.** Let $H_i \subset \mathbb{R}^n$, $i = 1, \dots, r$ denote a collection hyperplanes through 0 with defining equations $\alpha_i = 0$. Then

$A = \cup_i H_i$ is a (central) real hyperplane arrangement. It has a canonical Whitney stratification given by the strata $\big(\cap_{i \in I} H_i \backslash (\cup_{j \notin I} H_j)\big)$ for each subset $I \subseteq \{1, \dots, r\}$.

For each hyperplane H_i, we let P_i denote the closure of a nonempty union of connected components of $H_i \backslash (\cup_{j \neq i} H_j)$. Then, $V = \cup_i P_i$ will be called a *partial hyperplane arrangement*. Such a partial hyperplane arrangement has Zariski closure the corresponding hyperplane arrangement, which is real coherent by Proposition 2.7. Hence, it is a special semianalytic set by Proposition 2.6. A sample of model semianalytic sets which model geometric features in [DGH] are given in Figure 1.

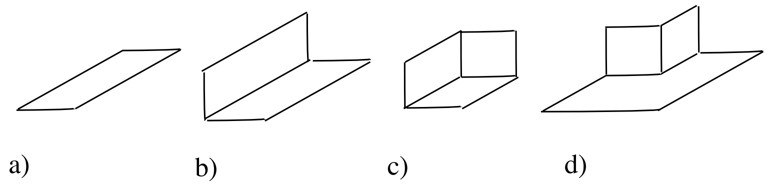

FIGURE 1. Examples of partial hyperplane arrangements which occur as models for feature stratifications: a) edge of surface; b) crease; c) convex or concave corner; and d) notch or saddle corner.

There are further examples which occur for generic structure of Blum medial axis which is the Maxwell set for the family of distance functions to the boundary hypersurface of a region, as in [M2] or [Y], are given in b) and c) in Figure 2.

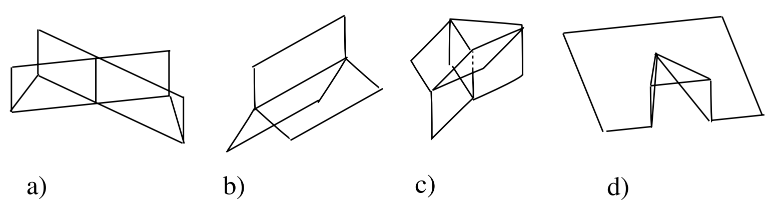

FIGURE 2. Examples of partial hyperplane arrangements which do not occur as models for feature stratifications: a) piecewise linear model of Whitney umbrella; b) and c) generic models for Blum medial axes; and d) nongeneric corner.

A second example involves 1-dimensional special semianalytic sets. First, $\mathbb{R}_+, 0 = \{x \in \mathbb{R} : x \geq 0\} \subset \mathbb{R}$ with its Whitney stratification is immediately seen to satisfy SSC. Hence, by 1) of Proposition 2.3, the image of $\mathbb{R}_+, 0$ under an analytic diffeomorphism satisfies SSC. Hence, a half-branch of a smooth semianalytic curve in an analytic submanifold satisfies SSC. More generally, a germ of a 1–dimensional semianalytic set in an analytic manifold which consists of branches or half-branches of smooth analytic curves satisfies the condition SSC (see Example 5.5 and Proposition 5.6 of [DGH, Chap. 5]). This yields the following.

Proposition 2.9. *A* 1*-dimensional semianalytic set* $V, 0 \subset \mathbb{R}^n, 0$ *consisting of irreducible branches of real analytic curves and half-branches of smooth analytic curves has a special semi-analytic stratification consisting of* $\{V \backslash \{0\}, \{0\}\}$.

Example 2.10 (Stratifications Refining Geometric Features by Shade/Shadows)**.** It follows from Proposition 2.4, that the refinement of a partial hyperplane arrangement by a 1-dimensional special semianalytic stratification is again a special semianalytic stratification, and hence semi-coherent. Using this result, it is proven in [DGH] that the stratifications resulting from the refinement of any stratification defining a generic geometric feature by the shade/shadow curves resulting from light in a generic direction is again a special semianalytic stratification $\mathcal{V}$ (and hence semi-coherent). This enabled the classification of (topologically) stable and (topological) codimension 1 germs for ${}_{\mathcal{V}}\mathcal{A}$-equivalence for each such stratification $\mathcal{V}$. The list of such stratifications and the corresponding classification of germs are given in Chapters 6, 7 and 8 of [DGH].

3. Equivalences of Mappings on Stratifications or Preserving Stratifications

We consider the groups of equivalences $\mathcal{G}_{\mathcal{V}}$ or ${}_{\mathcal{V}}\mathcal{G}$ preserving a stratification $\mathcal{V}$, defined by $V = V_k \supset V_{k-1} \supset \cdots \supset V_0 = \{0\}$, where in the holomorphic or real analytic category the stratification is holomorphic (the $(V_i, 0)$ are holomorphic germs), resp. real analytic (the $(V_i, 0)$ are real analytic germs) and in the smooth category it is a semi-coherent semianalytic stratification. To speak of all three of these categories, we denote the corresponding ring of germs by $\mathcal{C}_n$. We also let θ_n denote the module of germs of vector fields on $(\mathbf{k}^n, 0)$ in the appropriate category. We explain how these groups satisfy the conditions for being geometric subgroups of $\mathcal{A}$ or $\mathcal{K}$ and hence the basic theorems of singularity theory are valid for them. The explanation follows the same form as that for the case for $\mathcal{G}_V$ or ${}_V\mathcal{G}$ given in [D3, §8] and [D4, §9, 10].

${}_{\mathcal{V}}\mathcal{A}$ as a geometric subgroup.

We now carry out the explanation for the case of ${}_{\mathcal{V}}\mathcal{A}$-equivalence, with that for the other groups being analogous. Then, ${}_{\mathcal{V}}\mathcal{A}$ consists of the group of pairs of diffeomorphisms (h, h') (in the appropriate category) where $h : \mathbf{k}^n, 0 \to \mathbf{k}^n, 0$ and $h' : \mathbf{k}^p, 0 \to \mathbf{k}^p, 0$ with h preserving the strata of $\mathcal{V}$. This group is a subgroup of $\mathcal{A}$ and acts on germs $f_0 : \mathbf{k}^n, 0 \to \mathbf{k}^p, 0$ in the appropriate category by $(h, h') \cdot f_0 = h' \circ f_0 \circ h^{-1}$. There are corresponding unfolding groups acting on unfoldings. ${}_{\mathcal{V}}\mathcal{A}_{un}(q)$ consists of unfoldings of diffeomorphisms on q parameters (H, H') acting on unfoldings F on q parameters by $(H, H') \cdot F = H' \circ F \circ H^{-1}$.

We let Derlog$(\mathcal{V})$ be given by (2.1) for any of the three categories. In the holomorphic or real analytic categories, Derlog$(\mathcal{V})$ is a finitely generate module over $\mathcal{C}_n$ (denoting the ring of holomorphic, resp. real analytic germs). In the smooth category, it is finitely generated over $\mathcal{E}_n$ modulo infinitely flat vector fields. If (h_t, t) is a one-parameter group of unfoldings in the unfolding group $\mathcal{D}_{\mathcal{V},un}(1)$, then as h_t preserves the strata of $\mathcal{V}$, it follows that $\zeta = \frac{\partial h_t}{\partial t}_{|t=0}$ is tangent to the strata of $\mathcal{V}$, so $\zeta \in$ Derlog$(\mathcal{V})$. If h_t fixes 0, then ζ vanishes on 0, and belongs to Derlog$(\mathcal{V})^0$, the submodule of germs which vanish at 0. Conversely, the one-parameter subgroup h_t of germs of diffeomorphisms generated by some $\zeta \in$ Derlog$(\mathcal{V})$ will preserve the strata of $\mathcal{V}$. Hence, (h_t, t) is in the group of one-parameter unfoldings $\mathcal{D}_{\mathcal{V},un}(1)$. If in addition, h_t fixes 0, then ζ vanishes at 0, and conversely. Thus, the extended tangent space $T\mathcal{D}_{\mathcal{V},e} =$ Derlog$(\mathcal{V})$, with $T\mathcal{D}_{\mathcal{V}} =$ Derlog$(\mathcal{V})^0$ (the submodule of

Derlog($\mathcal{V}$) consisting of vector fields vanishing at 0). Thus, $T\,{}_{\mathcal{V}}\mathcal{A}_e$ can be written

$$T\,{}_{\mathcal{V}}\mathcal{A}_e \quad = \quad \mathrm{Derlog}(\mathcal{V}) \,\oplus\, \theta_p \tag{3.1}$$

Likewise, the tangent space $T\,{}_{\mathcal{V}}\mathcal{A}$ is given by

$$T\,{}_{\mathcal{V}}\mathcal{A} \quad = \quad \mathrm{Derlog}(\mathcal{V})^0 \,\oplus\, m_p \cdot \theta_p \tag{3.2}$$

For the smooth category, if $(\mathcal{V}, 0)$ is a semi-coherent semianalytic stratification of a closed semianalytic subset $V, 0 \subset \mathbb{R}^n, 0$, then by the results in §2, we may replace Derlog($\mathcal{V}$) by $\mathcal{E}_n\{\xi_1, \dots, \xi_m\}$ with ξ_j given in Definition 2.1. Then, the infinitesimal orbit map is the restriction of that for $\mathcal{A}$.

$$d\alpha_{f_0}(\xi, \eta) = \eta \circ f_0 \; - \xi(f_0) \qquad \text{for } \xi \in \mathrm{Derlog}(\mathcal{V}) \text{ and } \eta \in \theta_p \tag{3.3}$$

Then, just as for the case of ${}_V\mathcal{A}$, for f_0 in the appropriate category, $T\,{}_{\mathcal{V}}\mathcal{A}_e$ is a finitely generated module over the adequately ordered system of rings $f_0^* : \mathcal{C}_p \to \mathcal{C}_n$ (modulo infinitely flat vector fields in the smooth category), and $d\alpha_{f_0}$ would be a homomorphism of such modules. Hence, ${}_{\mathcal{V}}\mathcal{A}$ would satisfy the four conditions to be a geometric subgroup of $\mathcal{A}$ (the other three are easily seen to hold, using the modified version of the tangent space condition for the smooth category).

Hence, applying the results in [D2] and [D3], we conclude

Theorem 3.1. *Suppose $\mathcal{V}, 0$ is a stratification of $V, 0 \subset \mathbf{k}^n, 0$ of the corresponding type for each category of mappings: holomorphic, real analytic, or semi-coherent semianalytic stratification for the smooth category, then ${}_{\mathcal{V}}\mathcal{A}$ is a geometric subgroup of $\mathcal{A}$ (using (3.1) and (3.2)) for the adequately ordered system of rings $\{\mathcal{C}_n, \mathcal{C}_p\}$. Hence, both the finite determinacy and versal unfolding theorems and their consequences are valid for ${}_{\mathcal{V}}\mathcal{A}$.*

There is an analogous result for any ${}_{\mathcal{V}}\mathcal{G}$ or $\mathcal{G}_{\mathcal{V}}$ for $\mathcal{G} = \mathcal{A}, \mathcal{K}, \mathcal{R}$.

Example 3.2. The version of Theorem 3.1 for the case of special semianalytic stratifications is applied in [DGH] to the stratifications in $\mathbb{R}^3$ arising as refinements by shade/shadow curves of the stratifications by generic geometric features. The theorem together with application of classsification methods in [BKD], [BDW], and [Kr] and the topological methods in [D3] and [D4] yields the classification of both the (topologically) ${}_{\mathcal{V}}\mathcal{A}$-stable projections of the stratifications and the (topological) codimension 1 transitions given by Theorem 4.1 in Chap. 6 and Theorem 5.1 in Chap. 7 of [DGH].

$\mathcal{A}(\mathcal{V})$ as a geometric subgroup.

Let $\mathcal{V}$ be a stratification of a germ $(V, 0)$. Instead of $\mathcal{A}$-equivalence preserving a stratification $\mathcal{V}$, we may consider instead $\mathcal{A}$-equivalence for germs on $\mathcal{V}$, which we denote by the group $\mathcal{A}(\mathcal{V})$. For just the germ of a variety $(V, 0)$, the tangent space for the case of $\mathcal{A}(V)$ was determined in [D2, §8] and [D3, §9, 10]. To consider instead the germs on the stratification $\mathcal{V}$, the equivalence is defined via the group consisting of diffeomorphisms $H : \mathbf{k}^{n+p}, 0 \to \mathbf{k}^{n+p}, 0$, $h : \mathbf{k}^n, 0 \to \mathbf{k}^n, 0$, and $h' : \mathbf{k}^p, 0 \to \mathbf{k}^p, 0$, such that: i) $h \circ \pi_n = \pi_p \circ H$; ii) H preserves $V \times \mathbf{k}^p$; iii) $H|(V \times \mathbf{k}^p) = h \times h'$; and iv) h preserves the strata of $\mathcal{V}$. Then, $H \circ (h \times h')^{-1} \equiv id$ on $V \times \mathbf{k}^p$. A calculation then shows that

$$T\,\mathcal{A}(\mathcal{V})_e \quad = \quad \mathrm{Derlog}(\mathcal{V}) \,\oplus\, \theta_p \,\oplus\, I(V) \cdot \mathcal{C}_{n+p}\{\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_p}\}\,. \tag{3.4}$$

Likewise, the tangent space $T\,\mathcal{A}(\mathcal{V})$ is given by

$$T\,\mathcal{A}(\mathcal{V}) \quad = \quad \mathrm{Derlog}(\mathcal{V})^0 \oplus m_p \cdot \theta_p \oplus I(V) \cdot \mathcal{C}_{n+p}\{\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_p}\}\,. \tag{3.5}$$

Now the infinitesimal orbit map is defined by

$$d\alpha_{f_0}(\xi, \eta, \zeta) = \zeta \circ \tilde{f}_0 + \eta \circ f_0 \ - \xi(f_0) \tag{3.6}$$

where as above, $\xi \in \mathrm{Derlog}(\mathcal{V})$ and $\eta \in \theta_p$; in addition $\zeta \in I(V)\cdot\mathcal{C}_{n+p}\{\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_p}\}$, and $\tilde{f}_0(x) = (x, f_0(x))$.

Then, an analogous argument as above yields the following.

Theorem 3.3. *Suppose $\mathcal{V}, 0$ is a stratification of $V, 0 \subset \mathbf{k}^n, 0$ of the corresponding type for each category of mappings: holomorphic, real analytic, or semi-coherent semianalytic stratification for the smooth category, then $\mathcal{A}(\mathcal{V})$ is a geometric subgroup of $\mathcal{A}$ (using (3.4), (3.5)), and (3.5)) for the adequately ordered system of rings $\{\mathcal{C}_n, \mathcal{C}_p\}$. Hence, both the finite determinacy and versal unfolding theorems and their consequences are valid for $\mathcal{A}(\mathcal{V})$.*

Again there is an analogous result for $\mathcal{K}(\mathcal{V})$, and $\mathcal{R}(\mathcal{V})$.

Equivalences Allowing the Stratification to Deform.

Lastly, suppose that $(\mathcal{V}, 0)$ is defined as $g^{-1}(\mathcal{V}')$, for a stratification $\mathcal{V}'$ of a germ $V', 0 \subset \mathbf{k}^r, 0$, with the germ $g : \mathbf{k}^n, 0 \to \mathbf{k}^r, 0$ being finitely determined for $\mathcal{K}_{\mathcal{V}'}$-equivalence. Then, the equivalence of a germ $f : \mathbf{k}^n, 0 \to \mathbf{k}^p, 0$ on $(\mathcal{V}, 0)$, allowing both $\mathcal{V}$ and f to deform, is obtained by considering the action on the pair $(g, f) : \mathbf{k}^n, 0 \to \mathbf{k}^{r+p}, 0$ by $\mathcal{K}_{\mathcal{V}}$-equivalence on g and $\mathcal{A}$-equivalence on f, using a common diffeomorphism on $(\mathbf{k}^n, 0)$. Again, if the stratification $\mathcal{V}'$ is of the appropriate type for each category, then the equivalence group is a geometric subgroup of $\mathcal{A}$ or $\mathcal{K}$, and so the basic results of singularity theory apply for this equivalence.

Remark 3.4. We have concentrated on how the groups $\mathcal{G} = \mathcal{A}, \mathcal{K}, \mathcal{R}$ can be modified to allow an equivalence preserving a variety $(V, 0)$ or stratification $(\mathcal{V}, 0)$ for each of the three categories. In fact, for any geometric subgroup $\mathcal{G}$ which has a factor group $\mathcal{D}_r$, we can replace it by a subgroup $\mathcal{D}_V$ or $\mathcal{D}_{\mathcal{V}}$, for $V, 0 \subset \mathbf{k}^r, 0$ of $\mathcal{V}$ a stratification in $(\mathbf{k}^r, 0)$. Provided $(V, 0)$ or $(\mathcal{V}, 0)$ are appropriate for the category, the resulting group of equivalences will again be a geometric subgroup.

Concluding Remarks.

The local singularity-theoretic methods we have described apply to finite codimension germs for the appropriate equivalence group. The abundance of such germs will follow when the stratification $(\mathcal{V}, 0)$ or germ $(V, 0)$ is "holonomic"in the sense introduced by Saito [Sa]. By this we mean there is a neighborhood U of 0 such that for each $x \in U$, the generators $\{\xi_1, \dots, \xi_r\}$ of $\mathrm{Derlog}(\mathcal{V})$, resp. $\mathrm{Derlog}(V)$, span the tangent space $T_x S_i$ of the statum of $\mathcal{V}$, resp. the canonical Whitney stratification of $(V, 0)$, which contains x.

The special semianalytic stratifications which occur in [DGH] for the refinemments of the stratifications of geometric features by shade shadow curves are all holonomic. However, the classification shows that finite ${}_{\mathcal{V}}\mathcal{A}$-codimension germs of low codimension already are frequently multi-modal singularities; so that topological methods of [D3] and [D4] are needed to carry out the classification.

4. Proofs of the Results

It remains to prove the results concerning semi-coherence.

Proof of Proposition 1.6. First, for i), we let $f \in I(V)$. There exists a neighborhood $0 \in U \subset \mathbb{R}^n$ such that f is defined on U and vanishes on $V \cap U$. Also, we denote the weights of the coordinates on $\mathbb{R}^n$ by $\mathrm{wt}\,(x_i) = a_i > 0$ for $i = 1, \dots, n$. We expand the Taylor expansion of f in terms of weights $\hat{f}(x) = \sum_{j=1}^{\infty} f_j(x)$, where $\mathrm{wt}\,(f_j) = j$.

We claim that each $f_j \in I(V)$. If not, choose the smallest k for which this is not true. Suppose $x_0 \in V \cap U$ is such that $f_k(x_0) \neq 0$. Let $x_0 = (x_{0\,1}, \dots, x_{0\,n})$ and define $\gamma : \mathbb{R} \to \mathbb{R}^n$ by $\gamma(t) = (x_{0\,1} t^{a_1}, \dots, x_{0\,n} t^{a_n})$. By the weighted homogeneity of f_k, it follows $f_k \circ \gamma(t) = t^k f_k(x_0)$. Then, the Taylor expansion of $f \circ \gamma(t)$ is given by $\widehat{f \circ \gamma(t)} = \sum_{j=1}^{\infty} t^j f_j(x_0)$. On the one hand as $f \circ \gamma(t) = 0$ for $0 \leq t < \varepsilon$, the Taylor expansion of $f \circ \gamma(t)$ is zero. However, by assumption the coefficient of t^k is $f_k(x_0) \neq 0$, so it is the lowest nonzero term of the Taylor expansion, a contradiction. Thus, all $f_j \in I(V)$. As each f_j is analytic and $= 0$ on V, which has local analytic Zariski closure $\tilde{V}$, we conclude $f_j \in I^{an}(\tilde{V})$. Hence, we may write as a weighted homogeneous sum $f_j = \sum_{i=1}^{s} h_{i,j} g_i$, where g_i are a set of weighted homogeneous generators of $I^{an}(\tilde{V})$ with weights $\mathrm{wt}\,(g_i) = b_i > 0$. Hence, we may write as a formal sum

$$\hat{f} \;=\; \sum_{i=1}^{s} (\sum_{j=1}^{\infty} h_{i,j}) g_i \, .$$

As $\mathrm{wt}\,(h_{i,j}) = j - b_i$ the formal sum $\sum_{j=1}^{\infty} h_{i,j}$ defines an element $\hat{h}_i \in \mathbb{R}[[\mathbf{x}_n]]$, where $\mathbf{x}_n = (x_1, \dots, x_n)$.

Lastly, by Borel's Lemma, there is a germ $h_i \in \mathcal{E}_n$ with Taylor expansion $\hat{h}_i$. Thus, if we let $f' = \sum_{i=1}^{s} h_i g_i$, we have $\hat{f} = \hat{f}'$, or equivalently $f \equiv f' \bmod m_n^{\infty}$. As this holds for all $f \in I(V)$, the result i) follows.

For ii) we follow an analogous line of reasoning and use the same notation as for i). Let $\xi \in \mathrm{Derlog}(V)$. There is a neighborhood $0 \in U \subset \mathbb{R}^n$ so that both ξ and the generators g_j of $I^{an}(\tilde{I})$ are defined on U and so that (by Remark 1.1) $\xi(g_j)$ vanishes on $V \cap U$ for $j = 1, \dots, s$. We again consider a weighted expansion of the Taylor series of ξ, $\hat{\xi} = \sum_{j=n_0}^{\infty} \xi_j$, where ξ_j is weighted homogeneous of weighted degree j. Here, as usual, we assign weights $\mathrm{wt}\,(\dfrac{\partial}{\partial x_i}) = -a_i$ and then we let $n_0 = -\max_i\{a_i\}$.

We claim that each $\xi_j \in \mathrm{Derlog}^{an}(\tilde{V})$. If not let the lowest j for which this fails be denoted by k and for this k there is an g_ℓ so that $\xi_k(g_\ell)$ does not vanish on V in a neighborhood of 0, otherwise as it is analyic, it also vanishes on $\tilde{V}$, so $\xi_k(g_\ell) \in I^{an}(\tilde{V})$. If this held for each i, then $\xi_k \in \mathrm{Derlog}^{an}(\tilde{V})$. Hence, there is an $x_0 \in V \cap U$ so that $\xi_k(g_\ell)(x_0) \neq 0$. We consider the curve $\gamma(t)$ as above. Then $\xi(g_\ell)$ vanishes on $V \cap U$, and hence on the curve $\gamma(t)$ for $0 \leq t < \varepsilon$. Thus, the Taylor expansion of $\xi(g_\ell) \circ \gamma(t)$ is 0.

Then $\xi_j(g_\ell)$ is a weighted homogeneous polynomial of weighted degree $j + b_\ell > 0$ (if it is a nonzero polynomial). As we assume it is nonzero, we also have $\xi_j(g_\ell) \circ \gamma(t) = \xi_j(g_\ell)(x_0) t^{j+b_\ell}$. We then compute the Taylor expansion of $\xi(g_\ell) \circ \gamma(t)$ by

$$\xi(\widehat{g_\ell) \circ \gamma}(t) \;=\; \sum_{j=n_0}^{\infty} \xi_j(g_\ell)(x_0)\, t^{j+b_\ell}$$

Again, this Taylor series has a lowest nonzero term t^{k+b_ℓ}, contradicting that it is zero. Thus, each $\xi_j \in \mathrm{Derlog}^{an}(\tilde{V})$.

If by [Lo], $V = V_k \supset V_{k-1} \supset \cdots \supset V_1 \supset V_0 = \{0\}$ defines the canonical Whitney stratification $\mathcal{V}$, consisting of semianalytic sets (also invariant under $\mathbb{R}_+$), then we may apply the preceding argument to each V_i to conclude $\xi_j \in \mathrm{Derlog}^{an}(\tilde{V}_i)$. As ξ_j is tangent to the regular strata of each V_i, $\xi_j \in \mathrm{Derlog}^{an}(\mathcal{V})$, the submodule of $\mathrm{Derlog}^{an}(\tilde{V})$ consisting of germs of analytic vector fields tangent to the strata of $\mathcal{V}$.

As $\mathcal{A}_n$ is Noetherian, $\mathrm{Derlog}^{an}(\mathcal{V})$ is a finitely generated $\mathcal{A}_n$-module. As $\tilde{V}$, V, and $\mathcal{V}$ are invariant under the $\mathbb{R}_+$-action, $\mathrm{Derlog}^{an}(\mathcal{V})$ has a set of weighted homogeneous generators $\{\zeta_1, \dots, \zeta_r\}$ of weights $\mathrm{wt}\,(\zeta_j) = c_j$. We may write $\xi_j = \sum_{i=1}^{r} h_{i,j}\zeta_i$, where $h_{i,j}$ is weighted homogeneous of weighted degree $j - c_i$ (and $h_{i,j} = 0$ if $j - c_i < 0$). Thus, we may define $\hat{h}_i = \sum_{i=n_0}^{\infty} h_{i,j} \in \mathbb{R}[[\mathbf{x}_n]]$ and obtain

$$\hat{\xi} \;=\; \sum_{i=1}^{r} \hat{h}_i\zeta_i$$

Again, using Borel's lemma, there are smooth germs h_i whose Taylor expansions are $\hat{h}_i$, and we let $\xi' = \sum_{i=1}^{r} h_i\zeta_i$. We conclude $\xi \equiv \xi' \mathrm{mod}\ m^{\infty}\theta_n$. As this holds for every $\xi \in \mathrm{Derlog}(V)$, we obtain ii). □

Propositions 2.7 and 2.6 were proven in [DGH, Chap. 5]. Also, Propositions 2.3 and 2.4 were proven for the case of special semianalytic stratifications in [DGH, Chap. 5, §6]; however, the conditions i) and ii) in Definition 2.1 directly follow from the arguments given in the proofs for the special semianalytic case.

We do remark that to deal with the lack of weighted homogeneity which was used heavily in the proof of Proposition 1.6, the arguments proceed by first reducing to the formal category, and using the Artin approximation theorem and the Artin-Rees Lemma to obtain the desired generators there. Then, Borel's Lemma gives the desired result. These ideas are used repeatedly in the proofs in [DGH].

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