ON THE LEGACY OF FREE DIVISORS II: FREE* DIVISORS AND COMPLETE INTERSECTIONS

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To Vladimir Arnol'd who has had such an immense influence on mathematics

Introduction

The notion of free divisors first appeared in investigations of discriminants of versal unfoldings of isolated hypersurface singularities. They appeared implicitly in Arnol'd's work on wave front evolution via functions on discriminants of stable A_k singularities [A], and they were formally defined and investigated by Saito [Sa]. Since then, the notion of a free divisor has played a fundamental role for understanding the structure of nonisolated singularities such as discriminants, bifurcation sets, hyperplane arrangements, etc. As just one example, we mention that the results of Arnol'd [A3] and Brieskorn [B] on the factorization of the Poincaré polynomial for the homology of complement of a reflection hyperplane arrangement were shown by Terao [To1] to follow from the freeness of the arrangement. At the same time, the richness of the class of free divisors has also become increasingly evident, as researchers have established that a number of natural constructions yield free divisors (e.g. Saito [Sa], Looijenga [L], Terao [To1] [To2], Bruce [Br], Van Straten [VS], Mond [Mo3] [MVS], Goryunov [Go2], Grandjean [Gr] and this author [D6] (and see [D6] for other references).

Moreover, $V, 0$ being a free divisor allows us to investigate the vanishing topology (of singular Milnor fibers) of nonisolated singularities arising as nonlinear sections [DM], [D3], [D4]. This includes the higher multiplicities of V itself (which includes the topology of the complex link).

In the first part of this paper [D6], we considered how a free divisor $V, 0 \subset \mathbb{C}^p, 0$ passes on its freeness to divisors arising as $\mathcal{K}_V$–discriminants $D_V(F)$ for versal unfoldings F of nonlinear sections $f_0 : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$ of V. This general construction encompasses a number of the specific constructions mentioned above. For $D_V(F)$ to be a free divisor, it is sufficient that V generically has "Morse–type singularities" in dimension n where $n < hn(V)$, the holonomic codimension of V. This means that there are sections through points of V which exhibit many of the same properties as classical Morse singularities (appropriately understood). There are many basic examples which satisfy this condition. They include bifurcation sets for a certain well–defined class of finitely $\mathcal{A}$–determined complete intersection germs, smoothings of certain classes of isolated complete intersection singularities and space curve singularities, discriminants for generic hyperplane arrangements, etc.

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In this continuation of part I [D6] we address questions concerning the structure of discriminants and their vanishing topology for many different groups of equivalences involving nonisolated singularities. These include: nonisolated complete intersections defined as nonlinear sections of fixed complete intersections, singularities of functions, mappings, or divisors on nonisolated complete intersections, or equivalence of functions and mappings preserving a hypersurface V in the source (which can be viewed as boundary singularities for the singular boundary V in the sense of Arnol'd [A2], Siersma [Si], Lyashko [Ly], Bruce and Giblin [BG1] [BG2], and Tari [Ta], or singularities at infinity with V denoting the divisor at infinity).

We address three questions concerning discriminants for such equivalences.

(1) Is there a general criterion ensuring that for an equivalence group $\mathcal{G}$, the discriminants for $\mathcal{G}$–versal unfoldings are free divisors?
(2) If this criterion fails, is there a weaker substitute for the notion of free divisor which still generally applies?
(3) With this weaker notion, is it still possible to deduce results about the vanishing topology of nonlinear sections of such divisors?

Our goal in this paper is to give answers to these questions. First we give a criterion for freeness of discriminants which is described by the motto

$$\text{Cohen–Macaulay of codim } 1 + \text{Genericity of Morse Type Singularities} \Longrightarrow \text{Freeness of Discriminants}$$

We consider the class of geometrically defined subgroups $\mathcal{G}$ of $\mathcal{A}$ or $\mathcal{K}$ so the basic theorems of singularity theory are valid. We introduce the notion of $\mathcal{G}$ being *Cohen–Macaulay.* The geometric notion of "genericity of Morse–type singularities"implies a condition of "genericity of $\mathcal{G}$–liftable vector fields". Then, the above motto is given form by Theorem 1: $\mathcal{G}$ being Cohen–Macaulay with genericity of $\mathcal{G}$–liftable vector fields implies the freeness of discriminants for $\mathcal{G}$–versal unfoldings.

For the second question, we consider what remains true when either of these conditions fail. For example, for a free divisor $V, 0$, provided $n < hn(V)$ the group $\mathcal{K}_V$ is Cohen–Macaulay; but there are many examples where genericity of Morse–type singularities fails (see [D6]). However, in a very well–defined sense, the discriminants (and bifurcation sets) for these cases "just fail"to be free divisors. We are led to introduce the weaker notion of a "free* divisor"(pronounced "free star divisor"). A "free* divisor"structure is defined by a free submodule of vector fields which are tangent to the discriminant, but define the discriminant with a nonreduced structure. In this sense, the "*" denotes that the divisor is "free" with an asterisque indicating the lack of reduced structure.

Such a structure is highly nonunique (there is even a trivial free* divisor structure, which carries no new useful information). Hence, the importance of a particular free* divisor structure depends upon: first, its being defined in terms of natural properties of the divisor, and second the extent to which it can still be used to obtain useful topological and geometric information about the divisor.

In Theorem 2, we deduce that if $\mathcal{G}$ is only Cohen–Macaulay, then discriminants for $\mathcal{G}$–versal unfoldings are still free* divisors for the module of $\mathcal{G}$–liftable vector fields. We deduce Theorem 3 that for a free divisor V, even in the absence of genericity of Morse-type singularities, provided $n < hn(V)$ the $\mathcal{K}_V$–discriminants of versal unfoldings are always free* divisors for the module of $\mathcal{K}_V$–liftable vector fields. We conclude (Corollaries 2.17 – 2.19) that for finitely $\mathcal{A}$ determined germs in the nice dimensions, for complete intersection germs on ICIS, and for "almost free"nonlinear arrangements, the bifurcation sets are always free* divisors for the module of liftable vector fields.

These theorems are applied in part III of this paper to boundary singularities for singular boundary V. A standard group $\mathcal{G} = \mathcal{A}, \mathcal{R}^+, \mathcal{K}$ which also preserves V in the source is denoted ${}_V\mathcal{G}$. The equivalence group ${}_V\mathcal{K}$ is Cohen–Macaulay. Surprisingly whether ${}_V\mathcal{K}$ generically has Morse type singularities depends upon whether V does for $\mathcal{K}_V$–equivalence. Hence, Arnol'd [A2] and Lyashko [Ly] show that the simple boundary singularities of functions have versal unfoldings whose discriminants are free divisors. Our results show this fails in general; and when it holds it does so for functions and complete intersection mappings. This leads to the freeness of the discriminants for ${}_V\mathcal{A}$ stable or ${}_V\mathcal{R}^+$–versal germs and ${}_V\mathcal{A}$–bifurcation sets.

The second way the motto may fail is if the group $\mathcal{G}$ is not Cohen–Macaulay. It may still be possible that $\mathcal{G}$ –discriminants may still behave as if $\mathcal{G}$ were. We see this by introducing the notion of a *Cohen–Macaulay reduction* $\mathcal{G}^*$ of $\mathcal{G}$ (or more briefly a CM–reduction). We replace the group $\mathcal{G}$ by a subgroup $\mathcal{G}^*$ with better algebraic properties, but with the same discriminant for versal unfoldings. We prove as the second parts of Theorems 1 and 2 that if $\mathcal{G}$ has such a CM–reduction $\mathcal{G}^*$, then the $\mathcal{G}$–discriminants for versal unfoldings are free* divisors but for the module of $\mathcal{G}^*$–liftable vector fields. Moreover, if in addition "$\mathcal{G}$ generically has $\mathcal{G}^*$–liftable vector fields", then the $\mathcal{G}$–discriminant is a free divisor.

A basic situation where CM–reduction arises involves nonlinear sections of non-isolated complete intersections. We introduce the notion of "free complete intersection" which for nonisolated complete intersections is the analogue of the notion of free divisor (§5). For example, products of free divisors are free complete intersections. Nonlinear sections of a free complete intersection give the nonisolated analogue of ICIS. Moreover, complete intersections may possess Morse–type singularities with the analogous properties as for divisors. However, unexpectedly, even if a free complete intersection V generically has Morse–type singularities, the $\mathcal{K}_V$–discriminants for versal unfoldings of sections of V need not be free because generally $\mathcal{K}_V$ is not Cohen–Macaulay. We give an explicit example in §3 where this fails. Hence, the result of Looijenga [L] on the freeness of the discriminant for stable germs defining an ICIS does not extend to nonisolated complete intersections.

However, for free complete intersections $V, 0$, we establish (Theorem 4) that provided $n < h(V)$ (which can be smaller than $hn(V)$), $\mathcal{K}_V$ has a Cohen–Macaulay reduction $\mathcal{K}_V^*$. Thus, by theorem 2, $\mathcal{K}_V$–discriminants are free* divisors for the module of $\mathcal{K}_V^*$–liftable vector fields. In the case of an ICIS (corresponding to $V = \{0\} \subset \mathbb{C}^p$), the $\mathcal{K}_V^*$ and $\mathcal{K}_V$–liftable vector fields agree, and define the discriminant for the versal unfolding with reduced structure, recovering the result of Looijenga [L]. However, Theorem 1 does not generally apply to this case because for Morse–type singularities for $\mathcal{K}_V$, vector fields are not $\mathcal{K}_V^*$–liftable. We furthermore show in part III that the question of CM–reduction and genericity of Morse–type singularities reappear for the relative case of a divisor on a nonisolated complete intersection.

Lastly, we give an answer to the third question we raised regarding the vanishing topology of nonlinear sections of free* divisors. We show in §4 that the weaker properties of free* divisors are still sufficient to give formulas for the singular Milnor numbers for nonlinear sections (Theorem 5) in terms of codimensions of appropriate normal spaces. Except now we must take into account "virtual singularities" arising from the free* divisor structure. Because of the form for the normal spaces, these codimensions are also given as Buchsbaum–Rim multiplicities of the normal spaces, leading to natural questions about whether the formulas can be expressed in terms of Buchsbaum–Rim multiplicities of the original $\mathcal{G}$–normal spaces.

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CONTENTS

I Discriminants as Free or Free* Divisors and Their Vanishing Topology

1. Free* Divisors

Notation. We recall some of the notation used in §1 of Part I of this paper [D6]. If $(V, 0) \subset \mathbb{C}^p, 0$ is a germ of a variety, we let $I(V)$ denote the ideal of germs vanishing on V. We also let θ_p denote the module of vector fields on $\mathbb{C}^p, 0$ and consider the module of vector fields tangent to V.

$$\operatorname{Derlog}(V) = \{\zeta \in \theta_p : \zeta(I(V)) \subseteq I(V)\}.$$

Then, (following Saito [Sa]) if $V, 0$ is a hypersurface, it is called a *Free Divisor* if $\operatorname{Derlog}(V)$ is a free $\mathcal{O}_{\mathbb{C}^p,0}$–module. Its rank is then necessarily p. We further recall (see [DM]) that H is a *good defining equation* for V if there is an "Euler–like vector field" $e \in \operatorname{Derlog}(V)$ such that $e(H) = H$. This follows if V is weighted homogeneous, and H can be chosen weighted homogeneous of non-zero weight. However, we can always find a good defining equation by replacing V by $V \times \mathbb{C}$ (see [DM, §3] and (1.4) below). This causes no changes in the properties; if V is a free divisor then so is $V \times \mathbb{C}$. Also, by results in [D2], for the equivalence of sections of germs of varieties, we can replace V by $V \times \mathbb{C}$ without changing the deformation theory. Finally if M is a submodule of θ_p generated by $\{\zeta_1, \dots, \zeta_r\}$, we let $\langle M\rangle_{(y)}$ be the subspace of $T_y\mathbb{C}^p$ spanned by $\{\zeta_{1(y)}, \dots, \zeta_{r(y)}\}$.

We now introduce a weakened form of the notion of free divisor.

Definition 1.1. For a hypersurface germ $V, 0 \subset \mathbb{C}^p, 0$, and $p' = p + m \geq p$, let $V' = V \times \mathbb{C}^m \subset \mathbb{C}^{p'}$. Then, a *free* divisor structure for* V defined on $\mathbb{C}^{p'}$, consists of an $\mathcal{O}_{\mathbb{C}^{p'},0}$–submodule $\mathcal{L} \subseteq \mathrm{Derlog}(V')$ which satisfies:

(1) $\mathcal{L}$ is a free $\mathcal{O}_{\mathbb{C}^{p'},0}$–module of rank p'; and
(2) $\mathrm{supp}(\theta_{p'}/\mathcal{L}) = V'$

Although a free* divisor structure for V is not unique, frequently $\mathcal{L}$ will be a close approximation to $\mathrm{Derlog}(V)$. Hence, we will denote $\mathcal{L}$ by $\mathrm{Derlog}^*(V)$ and refer to $V, 0$ as *a free* divisor defined by* $\mathrm{Derlog}^*(V)$.

Remark 1.2. This notion is really a "stable property"of V as the free* divisor structure exists on $V' = V \times \mathbb{C}^m$ rather than on V itself. However, as remarked earlier, we can replace V by V' without changing either the deformation theory or vanishing topology for nonlinear sections. Furthermore, many cases we consider involve discriminants and bifurcation sets for versal unfoldings, so we may replace them by their products with some $\mathbb{C}^m$. Hence, in what follows, to reduce notation, we shall frequently suppose that we have already replaced V by its product with the appropriate $\mathbb{C}^m$ so that $\mathrm{Derlog}^*(V) \subseteq \mathrm{Derlog}(V)$.

Remark 1.3. Unlike $\mathrm{Derlog}(V)$, $\mathrm{Derlog}^*(V)$ need not be a Lie algebra of vector fields. However, frequently V will denote the discriminant (or bifurcation set) for a versal unfolding with respect to some geometric subgroup $\mathcal{G}$ of $\mathcal{A}$ or $\mathcal{K}$ [D5]. Then, $\mathrm{Derlog}^*(V)$ will be the extended tangent space to the group of $\mathcal{G}$–liftable diffeomorphisms, which is a subgroup of $\mathcal{D}_V$, the diffeomorphisms preserving V. Hence, $\mathrm{Derlog}^*(V)$ will naturally have a Lie algebra structure. It provides V with a singular foliation. Thus, when $\mathrm{Derlog}^*(V)$ does have this additional Lie algebra structure, we will specifically note it.

(1.4) Properties of Free Divisors.* 1) As for free divisors, we say that h is a *good defining equation* for $V, 0$ if h is reduced and there is an Euler–like vector field $e \in \mathrm{Derlog}^*(V)$ satisfying $e(h) = h$. By replacing V by $V' = V \times \mathbb{C}$ and h by $h_1(x,t) = exp(t) \cdot h$, we obtain an Euler–like vector field $e = \frac{\partial}{\partial t}$. Again $V', 0$ has a free*–divisor structure with

$$\mathrm{Derlog}^*(V') = \mathcal{O}_{\mathbb{C}^{p+1},0}\{\zeta_1, \dots, \zeta_p, \frac{\partial}{\partial t}\}$$

where $\{\zeta_1, \dots, \zeta_p\}$ generate $\mathrm{Derlog}^*(V)$, with good defining equation h_1. In all that follows, we will see that this argument allows us to assume that the natural modules of vector fields defining free* divisors may always be assumed to contain an Euler–like vector field. We will refer to the above as the *standard construction of an Euler–like vector field.*

2) For a free* divisor $V, 0 \subset \mathbb{C}^p$ with good defining h and Euler–like vector field e we let

$$\mathrm{Derlog}^*(h) = \{\zeta \in \mathrm{Derlog}^*(V) : \zeta(h) = 0\}$$

By the same argument as in [DM, §2], $\mathrm{Derlog}^*(h)$ is a free $\mathcal{O}_{\mathbb{C}^p,0}$–module of rank $p-1$ and

$$\mathrm{Derlog}^*(V) = \mathrm{Derlog}^*(h) \oplus \mathcal{O}_{\mathbb{C}^p,0}\{e\}$$

Finally, as for free divisors, we define $T_{log^*}(V)_{(y)} = \langle \mathrm{Derlog}^*(V) \rangle_{(y)}$ and similarly for $T_{log^*}(h)_{(y)}$.

3) *Measuring Nontriviality for Free* Divisors*

For any hypersurface $V, 0 \subset \mathbb{C}^p$ with reduced defining equation h, one free* divisor structure is defined using $\mathrm{Derlog}^*(V) = h \cdot \theta_p$, which is trivial in the sense that for any $y \in V$, we have $T_{log^*}(V)_{(y)} = (0)$.

Such trivial free* divisor structures provide no new useful infomation. The value of a free* divisor structure depends on how closely it approximates $\mathrm{Derlog}(V)$. We next define geometric measures of this closeness. For any $\mathcal{O}_{\mathbb{C}^p,0}$–module $M \subseteq \theta_p$ we let $\mathcal{F}_q(\theta_p/M)$ denote the q–th Fitting ideal of θ_p/M and let $F_q(M) = supp(\mathcal{F}_q(\theta_p/M))$. Then, $F_{p-q}(M) = \{y \in \mathbb{C}^p : \dim_{\mathbb{C}}\langle M\rangle_{(y)} < q\}$. If $F_{p-q-1}(M) \neq F_{p-q}(M)$ and $\mathrm{codim}(F_{p-q}(M)) = s$, then we say M *has rank (at least) q off codimension s.*

For specific modules of vector fields, we abbreviate $F_q(\mathrm{Derlog}^*(V))$ by $F_q^*(V)$, $F_q(\mathrm{Derlog}^*(h))$ by $F_q^*(h)$, and similarly for $F_q(V)$ using instead $\mathrm{Derlog}(V)$, etc. As $\mathrm{Derlog}^*(V) \subseteq \mathrm{Derlog}(V)$, $F_q^*(V) \subseteq F_q(V)$. In the case that $\mathrm{Derlog}^*(V)$ is a Lie subalgebra of $\mathrm{Derlog}(V)$, $\mathrm{Derlog}^*(h)$ is a Lie ideal. Both $\mathrm{Derlog}^*(V)$ and $\mathrm{Derlog}^*(h)$ define singular foliations of V. Then, $F_{p-q}^*(h) \subseteq F_{p-q}^*(V)$ define the analytic subsets where the leaves of the foliations have dimension $< q$.

We shall see in §4 that to use the free* divisor structure to compute the number of singular vanishing cycles for nonlinear sections $f_0 : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$ of $V, 0$, we must have f_0 transverse off 0 to the singular distribution of V defined by $\mathrm{Derlog}^*(h)$. For example, if $n < p$, then for each $0 \leq q < p - n$, this requires $n < \mathrm{codim}(F_{p-q}^*(h))$.

By the *reduced rank* of the free* structure at a point $y \in V$, we shall mean $\dim_{\mathbb{C}} T_{log^*}(h)_{(y)}$. Then, $r = \dim_{\mathbb{C}} T_{log^*}(h)_{(0)}$ is the minimum reduced rank. If $n < p - r$, then a necessary condition for transversality off 0 is that the set of points where the reduced rank $= r$ has codimension $< n$. Thus, we seek free* divisor structures with as large a rank off as small a codimension as possible, with especially $F_{p-r}^*(h)$ as small as possible.

4) *Exponents for Free* Divisors*

Let $\{\zeta_1, \dots, \zeta_p\}$ generate $\mathrm{Derlog}^*(V)$ with $\zeta_i = \sum_j a_{ij} \dfrac{\partial}{\partial x_i}$. We denote $\det(a_{ij})$ by $\det(\zeta_1, \dots, \zeta_p)$, or more simply $\det(\zeta_i)$. It is a generator of $\mathcal{F}_0(\theta_p/\mathrm{Derlog}^*(V))$. Also, let h be a reduced defining equation for $V, 0$ with $h = \prod h_i$, where each h_i defines an irreducible component V_i of V. Then, by condition 2 of (1.1), $\det(a_{ij}) = u \cdot \prod h_i^{m_i}$, where u is a unit. The *exponents* m_i measure componentwise how much $\mathrm{Derlog}^*(V)$ fails to define a free divisor structure for V.

Remark 1.4. Many of the basic properties of free divisors given in [D3, §1- 3] extend to free* divisors. To extend both the definitions and results to free* divisors, we simply replace $T_{log}V$ and $T_{log}(h)$ by $T_{log^*}V$ and $T_{log^*}(h)$, and give the "*"analogues of algebraic transversality, algebraic general position, transverse union, almost free divisor, etc.

We illustrate the preceding with several examples.

Example 1.5. The Whitney umbrella $V, 0$ is the image of the finite map germ $f_0 : \mathbb{C}^2, 0 \to \mathbb{C}^3, 0$ with $f_0(x_1, x_2) = (x_1^2, x_1 x_2, x_2) = (Y, Z, W)$ and is defined by $H(Y, Z, W) = YW^2 - Z^2 = 0$. It is not a free divisor; in fact (see e.g. [D2, (1.1)], Derlog(V) is generated by

$$\text{(1.1)}\quad \eta_1 = 2Y\frac{\partial}{\partial Y} + Z\frac{\partial}{\partial Z}, \quad \eta_2 = 2Y\frac{\partial}{\partial Y} - W\frac{\partial}{\partial W}, \quad \eta_3 = WY\frac{\partial}{\partial Z} + Z\frac{\partial}{\partial W},$$
$$\text{and} \qquad \eta_4 = 2Z\frac{\partial}{\partial Y} + W^2\frac{\partial}{\partial Z}.$$

However, it has several nontrivial free* divisor structures. One is given by D_1 generated by $\{e = a\eta_1 - b\eta_2, \eta_3, Z\eta_4 - W^2\eta_2\}$, where $a \neq b$, and a second by D_2 generated by $\{e = a\eta_1 - b\eta_2, \eta_4, Z\eta_3 + WY\eta_2\}$, where $a \neq b, 0$. D_1 has reduced rank 1 off the "handle", i.e. the Y–axis, while D_2 has reduced rank 1 off the Y and W–axes. Both have exponent 2. We use D_1 to compute the vanishing topology for nonlinear sections in §4.

Example 1.6. An arrangement of more than p hyperplanes through $0 \in \mathbb{C}^p$ which are in general position off 0 is not a free divisor for $p > 2$. However, such an arrangement is an almost free divisor, obtained as the pullback of a Boolean arrangement [D3, §5]. Consider the case of such an arrangement of four planes through $0 \in \mathbb{C}^3$. Any two such are equivalent by a linear transformation so we consider A defined by $xyz\ell = 0$ where $\ell = x + y + z$. Although it is not free, it is a free* divisor in $\mathbb{C}^4$ defined by Derlog*(A) with generators

(1.2)

$$\zeta_1 = x(z+\ell)\frac{\partial}{\partial x} - z(x+\ell)\frac{\partial}{\partial z} + \frac{\partial}{\partial w}, \quad \zeta_2 = -y(z+\ell)\frac{\partial}{\partial y} + z(y+\ell)\frac{\partial}{\partial z} + \frac{\partial}{\partial w},$$
$$\zeta_3 = -x(y+\ell)\frac{\partial}{\partial x} + y(x+\ell)\frac{\partial}{\partial y} + \frac{\partial}{\partial w}, \quad \text{and} \quad \zeta_0 (= e) = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}.$$

A computation shows $\det(\zeta_i) = xyz\ell^2$, so the free* structure is reduced on all hyperplanes except $\ell = 0$ and has reduced rank 2 up to codimension 3.

Third, consider general surface singularities in $V, 0 \subset \mathbb{C}^3, 0$. Surface singularities which are free divisors always have nonisolated singularities. Isolated surface singularities in $\mathbb{C}^3$ are "almost free divisors"in the sense of [D3]. Weighted homogeneous surface singularities with smooth singular set can be free or not, depending on special algebraic properties identified in [D8]. The next result shows that all weighted homogeneous surface singularities are nontrivial free* divisors.

Proposition 1.7. *If $V, 0 \subset \mathbb{C}^3$ is a weighted homogeneous surface singularity of nonzero weight, then $V, 0$ has a natural "Pfaffian" free* divisor structure in $\mathbb{C}^4$ of exponent 2 and of reduced rank 2 off Sing(V). In particular, if f has an isolated singularity, the free* divisor structure has reduced rank 2 off codimension 3.*

Proof. Let h be a weighted homogeneous defining equation for $V, 0$. We suppose $\text{wt}(h) = d \neq 0$, where $\text{wt}(y_i) = a_i$. Also, we denote the determinantal vector field $\eta_{ij} = \frac{\partial h}{\partial y_i}\frac{\partial}{\partial y_j} - \frac{\partial h}{\partial y_j}\frac{\partial}{\partial y_i}$. For $V \times \mathbb{C} \subset \mathbb{C}^4$, let w denote the coordinate for the last factor. We define the generators for Derlog*(V) by

$$\zeta_i \;=\; \eta_{jk} + a_i y_i \frac{\partial}{\partial w} \quad i = 1,2,3 \qquad \text{and} \qquad \zeta_0 \;=\; -de \;=\; -\sum a_i y_i \frac{\partial}{\partial y_i}.$$

where for $i > 0$, (ijk) is a cyclic permutation of (123).

The matrix of coefficients for $\zeta_1, \zeta_2, \zeta_3, \zeta_0$ is skew–symmetric; hence, its determinant is the square of the Pfaffian. A straightforward computation using the Euler relation for h shows the Pfaffian $= d \cdot h$, hence $\{\zeta_i\}$ defines a free* divisor structure of exponent 2. Furthermore, $\zeta_1, \zeta_2, \zeta_3$ generate $\mathrm{Derlog}^*(h)$ and their 2×2 minors generate an ideal containing $J(h)^2$. Thus, the free* structure has reduced rank 2 off $\mathrm{Sing}(V)$. $\square$

2. Cohen-Macaulay Properties of Geometric Subgroups

We now wish to extend the main Theorem 1 of Part 1 to more general situations involving discriminants for versal unfoldings for general geometric subgroups of $\mathcal{A}$ or $\mathcal{K}$. These groups include all of the standard equivalences. Moreover they satisfy the basic theorems of singularity theory such as the finite determinacy theorem, the versal unfolding theorem, and infinitesimal stability implies stability under deformations (see [D5]).

We consider generally a geometric subgroup $\mathcal{G}$ of $\mathcal{A}$ or $\mathcal{K}$ for the category of holomorphic germs . Recall this means that there is an action of $\mathcal{G}$ on $\mathcal{F}$ where $\mathcal{F}$ is an affine subspace of $\mathcal{C}(n,p)$, the space of holomorphic germs $f : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$. There is also a corresponding action of the group of unfoldings $\mathcal{G}_{un}(q)$ on the space of unfoldings $\mathcal{F}_{un}(q)$ on q parameters, for all integer $q \geq 0$. These actions satisfy four conditions given in [D5]. For an unfolding $F \in \mathcal{F}_{un}(q)$, we have the orbit map $\alpha_F : \mathcal{G}_{un}(q) \to \mathcal{F}_{un}(q)$ and the corresponding infinitesimal orbit map

$$d\alpha_F : T\mathcal{G}_{un,e}(q) \to T\mathcal{F}_{un,e}(q).$$

The extended tangent space is the image $d\alpha_F(T\mathcal{G}_{un,e}(q)) = T\mathcal{G}_{un,e} \cdot F$, and the normal space is the quotient $N\mathcal{G}_{un,e} \cdot F = T\mathcal{F}_{un,e}(q)/T\mathcal{G}_{un,e} \cdot F$. In the case of germs $f \in \mathcal{F}$, we denote the extended normal space by $N\mathcal{G}_e \cdot f$. These are analogues in the Thom–Mather framework of "(relative) T^1". A germ f has finite $\mathcal{G}$–codimension if $\dim_{\mathbb{C}} N\mathcal{G}_e \cdot f < \infty$. In this case it follows by an extension of the preparation theorem for adequate systems of rings [D5, Cor 6.16] that $N\mathcal{G}_{un,e} \cdot F$ is a finitely generated $\mathcal{O}_{\mathbb{C}^q,0}$–module.

We are interested in a special class of such subgroups $\mathcal{G}$. Let $F \in \mathcal{F}_{un}(q)$ be an unfolding of a germ f having finite $\mathcal{G}$–codimension. In what follows, we either represent F by $F(x,u) = (\bar{F}(x,u), u)$ or $(F_u(x), u)$. We are interested in values u such that $\bar{F}(\cdot, u)$ is not $\mathcal{G}$–stable. Of course, in general $\mathcal{G}$–equivalence does not necessarily make sense for germs at points other then 0. Nonetheless we can define the $\mathcal{G}$*–discriminant* of the unfolding F by

$$D_{\mathcal{G}}(F) \quad = \quad \mathrm{supp}(N\mathcal{G}_{un,e} \cdot F)$$

Following Teissier [Te], $D_{\mathcal{G}}(F)$ has an analytic structure given by the 0–th Fitting ideal of $N\mathcal{G}_{un,e} \cdot F$ (as an $\mathcal{O}_{\mathbb{C}^q,0}$–module).

Remark . There is ambiguity between what constitutes bifurcation sets versus discriminants for geometric subgroups. We should strictly speaking refer to $D_{\mathcal{G}}(F)$ as the bifurcation set of F. However, by [D9], it often can be viewed as the discriminant of an associated group $\mathcal{K}_V$ for appropriate V. For this reason we will think of discriminants and bifurcation sets as being examples for different geometric subgroups of this common notion of "discriminant". Where we want to specifically identify the bifurcation aspect we will refer to it as the bifurcation set.

If $D_{\mathcal{G}}(F)$ geometrically captures where $\bar{F}(\cdot, u)$ is not stable, this would imply that $D_{\mathcal{G}}(F)$ is preserved under $\mathcal{G}$ self–equivalences of F. Such equivalences are given infinitesimally by $\mathcal{G}$–liftable vector fields.

Definition 2.1. For a geometric subgroup $\mathcal{G}$ of $\mathcal{A}$ or $\mathcal{K}$, a *$\mathcal{G}$–liftable vector field for F* is a germ of a vector field $\eta \in \theta_q$ such that there is a germ of a vector field $\xi \in T\mathcal{G}_{un,e}$ satisfying:

$$d\alpha_F(\eta + \xi) = 0 \tag{2.1}$$

If we integrate $\eta + \xi$ and η, we obtain flows Φ and φ which commute with the projection $pr : \mathbb{C}^{n+p+q} \to \mathbb{C}^q$. If φ preserves $D_{\mathcal{G}}(F)$, then differentiating implies that $\eta \in \mathrm{Derlog}(D_{\mathcal{G}}(F))$. We formalize this property as follows.

Definition 2.2. We say that a geometric subgroup $\mathcal{G}$ of $\mathcal{A}$ or $\mathcal{K}$, has *geometrically defined discriminants* if for any versal unfolding F of a germ f having finite $\mathcal{G}$–codimension, if ξ is a $\mathcal{G}$–liftable vector field for F, then $\xi \in \mathrm{Derlog}(D_{\mathcal{G}}(F))$.

All geometric subgroups which we consider will have an appropriate geometric characterization of their discriminants and so have geometrically defined discriminants in this sense.

Example 2.3. For a germ of a variety $V, 0 \subset \mathbb{C}^p$, the group $\mathcal{K}_V$ has geometrically defined discriminants.

To see this let $F : \mathbb{C}^{n+q}, 0 \to \mathbb{C}^{p+q}, 0$ be a versal unfolding of a nonlinear section f_0 of V. Also, let $D_V(F)$ denote the $\mathcal{K}_V$–discriminant of F. We will show that if ζ is $\mathcal{K}_V$–liftable with local flow φ_t, then φ_t preserves $D_V(F)$. This implies $\zeta \in \mathrm{Derlog}(V)$.

Then, ζ being $\mathcal{K}_V$–liftable means there are germs of vector fields

$$\xi \in \mathcal{O}_{\mathbb{C}^{n+q},0}\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\} \quad \text{and} \quad \eta \in \mathcal{O}_{\mathbb{C}^{n+q},0}\{\zeta_1, \dots, \zeta_p\}$$

satisfying:

$$(\xi + \zeta)(\bar{F}) \quad = \quad \eta \circ \bar{F}. \tag{2.2}$$

Let φ_t, φ_{1t}, and ψ_t denote the flows induced by ζ, ξ, and η on respectively $\mathbb{C}^q, 0$, $\mathbb{C}^{n+q}, 0$ and $\mathbb{C}^{n+p+q}, 0$. Then, $\pi \circ \varphi_{1t} = \varphi_t \circ \pi$, for $\pi : \mathbb{C}^{n+q} \to \mathbb{C}^q$ the projection. Also, we have the commutative diagram

$$\begin{array}{ccc} \mathbb{C}^{n+q}, 0 & \xrightarrow{\tilde{F}} & \mathbb{C}^{p+q+n}, 0 \\ {\scriptstyle \varphi_{1t}} \uparrow & & {\scriptstyle \psi_t} \uparrow \\ \mathbb{C}^{n+q}, 0 & \xrightarrow{\tilde{F}} & \mathbb{C}^{p+q+n}, 0 \end{array} \tag{2.3}$$

where $\tilde{F}(x, u) = (\bar{F}(x, u), u, x)$. Then, the algebraic transversality of $\bar{F}(\cdot, u_0)$ to V at x_0 is equivalent to that of $\tilde{F}$ to $\tilde{V} = V \times \mathbb{C}^{q+n}$ at (x_0, u_0). This latter condition is equivalent to

$$d\tilde{F}(x_0, u_0)(T\mathbb{C}^{n+q}) + T_{log}\tilde{V}_{\tilde{y}_0} \quad = \quad T_{\tilde{y}_0}\mathbb{C}^{p+q+n}. \tag{2.4}$$

for $\tilde{y}_0 = \tilde{F}(x_0, u_0)$. However, as both φ_{1t} and ψ_t are diffeomorphisms for fixed t, it follows from (2.3) that by applying $d\psi_t(\tilde{y}_0)$, (2.4) is equivalent to

$$d\tilde{F}(x_0, u_0)(T\mathbb{C}^{n+q}) + d\psi_t(\tilde{y}_0)(T_{log}\tilde{V}_{\tilde{y}_0}) \quad = \quad T_{\tilde{y}_0}\mathbb{C}^{p+q+n}. \tag{2.5}$$

Finally, $\psi_t(\tilde{V}) = \tilde{V}$. Hence, let $\chi \in \mathrm{Derlog}(V)$ and let h define V. Then $\psi_{t*}(\chi)(h) = \chi(h \circ \psi_t)$. Since $h \circ \psi_t$ vanishes on $\tilde{V}$, it has the form $g(y, u, x) \cdot h$. Thus, $\chi(h \circ \psi_t) = g_1(y, u, x) \cdot h$. Hence, $\psi_{t*}(\chi) \in \mathrm{Derlog}(\tilde{V})$; and so $d\psi_t(\tilde{y}_0)(T_{log}\tilde{V}_{\tilde{y}_0}) = T_{log}\tilde{V}_{\psi_t(\tilde{y}_0)}$. Thus, transversality, or failure thereof, is preserved under the action by (φ_{1t}, ψ_t), and thus so is the $\mathcal{K}_V$–discriminant under φ_t. Thus, ζ is tangent to $D_V(F)$ and so belongs to Derlog(V). This completes the verification.

Now we are ready to introduce the Cohen–Macaulay property for geometric subgroups and state what we mean by a group having a reduction to such a group.

Definition 2.4. A geometric subgroup $\mathcal{G}$ of $\mathcal{A}$ or $\mathcal{K}$ will be said to be *Cohen–Macaulay* if it has geometrically defined discriminants; and if for each versal unfolding $F \in \mathcal{F}_{un}(q)$ of a germ f having finite $\mathcal{G}$–codimension, $N\mathcal{G}_{un,e} \cdot F$ is Cohen–Macaulay as a $\mathcal{O}_{\mathbb{C}^q,0}$–module, with supp$(N\mathcal{G}_{un,e} \cdot F)$ of dimension $q-1$.

Example 2.5. For a free divisor $V, 0 \subset \mathbb{C}^p, 0$ with $n < hn(V)$, the group $\mathcal{K}_V$ acting on nonlinear sections $f_0 : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$ is Cohen–Macaulay. To see this, we note that by Example 2.3, $\mathcal{K}_V$ has geometrically defined discriminants (this does not depend on V being free). Also, as $n < hn(V)$, by the proof of Lemma 3.4 of Part I [D6], $N\mathcal{K}_{V,un,e} \cdot F$ is Cohen–Macaulay and by Proposition 2.4 of [D6], $D_V(F) = \text{supp}(N\mathcal{K}_{V,un,e} \cdot F)$ has dimension $q-1$.

For complete intersections, the situation becomes more complicated. However, the analogous group is close to being Cohen–Macaulay. This can be made precise using the following definition.

Definition 2.6. Given a geometric subgroup $\mathcal{G}$ of $\mathcal{A}$ or $\mathcal{K}$ which has geometrically defined discriminants, a *Cohen–Macaulay reduction* of $\mathcal{G}$ (abbreviated CM–reduction) consists of a geometric subgroup $\mathcal{G}^* \subset \mathcal{G}$ which still acts on $\mathcal{F}$ (and $\mathcal{F}_{un}$) such that:

(1) $\mathcal{G}^*$ is Cohen–Macaulay;
(2) $f \in \mathcal{F}$ has finite $\mathcal{G}^*$–codimension iff it has finite $\mathcal{G}$–codimension;
(3) if $F \in \mathcal{F}_{un}(q)$ is an unfolding of a germ f which has finite $\mathcal{G}$–codimension, then viewed as $\mathcal{O}_{\mathbb{C}^q,0}$–modules,

$$\text{supp}(N\mathcal{G}^*_{un,e} \cdot F) \quad = \quad \text{supp}(N\mathcal{G}_{un,e} \cdot F). \tag{2.6}$$

Remark 2.7. It follows that the CM–reduction $\mathcal{G}^*$ also has geometrically defined discriminants. As $\mathcal{G}^*$ is a subgroup of $\mathcal{G}$, $\mathcal{G}^*$–liftable vector fields are also $\mathcal{G}$–liftable. Hence, they belong to Derlog$(D_{\mathcal{G}}(F)) = $ Derlog$(D_{\mathcal{G}^*}(F))$, by (2.3).

The second criterion in the "motto"stated in the introduction concerns the genericity of Morse-type singularities. Because each group $\mathcal{G}$ would require its own form of Morse-type singularity, we state the condition in a form which avoids considering individual cases.

Definition 2.8. We say that a geometric subgroup $\mathcal{G}$ of $\mathcal{A}$ or $\mathcal{K}$ with a CM-reduction $\mathcal{G}^*$, has *generically $\mathcal{G}^*$–liftable vector fields* if for a $\mathcal{G}$–versal unfolding F (on q parameters) of a germ f_0 of finite $\mathcal{G}$–codimension, there is a Zariski open subset Z of $D_{\mathcal{G}}(F)_{reg}$ (intersecting each component in a neighborhood of 0) such that:

(1) if h is a reduced defining equation for $D_{\mathcal{G}}(F)$ then there is an inclusion of sheaves restricted to Z

$$h \cdot \Theta_q \subset \mathit{Derlog}^*(D_{\mathcal{G}}(F));$$

where *Derlog**$(D_{\mathcal{G}}(F))$ denotes module of $\mathcal{G}^*$–liftable vector fields; and

(2) if $u \in Z$, then

$$T_{log^*}(D_{\mathcal{G}}(F))_{(u)} \quad = \quad T_u(D_{\mathcal{G}}(F)).$$

If $\mathcal{G}$ is itself Cohen-Macaulay, we say it has *generically $\mathcal{G}$–liftable vector fields* provided the preceeding conditions hold with $\mathcal{G}^* = \mathcal{G}$.

Remark . In part I, we showed that genericity of Morse type singularities for $\mathcal{K}_V$–equivalence for $V, 0$ a free divisor implied the genericity of $\mathcal{K}_V$–liftable vector fields. The crucial point is to be able to use the genericity of Morse–type singularities to establish the genericity of $\mathcal{G}^*$–liftable vector fields. We provide a strategy for doing this and use this several times in Part III [D7].

We can now state the first basic result which implies that discriminants for various geometric subgroups $\mathcal{G}$ are free divisors.

Theorem 1. *i) Suppose that the geometric subgroup $\mathcal{G}$ is Cohen–Macaulay and generically has $\mathcal{G}$–liftable vector fields, then $D_{\mathcal{G}}(F)$ is a free divisor with*

$$\mathrm{Derlog}(D_{\mathcal{G}}(F)) \quad = \quad \textit{module (Lie algebra) of } \mathcal{G}\textit{–liftable vector fields.}$$

ii) If instead $\mathcal{G}$ has a Cohen–Macaulay reduction $\mathcal{G}^*$ which generically has $\mathcal{G}^*$–liftable vector fields. Then, $D_{\mathcal{G}}(F)$ is a free divisor with

$$\mathrm{Derlog}(D_{\mathcal{G}}(F)) \quad = \quad \textit{module (Lie algebra) of } \mathcal{G}^*\textit{–liftable vector fields.}$$

As an indication of its applicability, we mention that besides including the results from part I, this theorem also applies to: groups of equivalences for functions and almost free divisors on an ICIS, allowing both to vary (by giving a CM–reduction), and to the group ${}_V\mathcal{K}$ for "boundary singularities", which is Cohen-Macaulay.

In the absence of genericity of Morse-type singularities, we can still conclude

Theorem 2. *i) Suppose that the geometric subgroup $\mathcal{G}$ is Cohen–Macaulay. Then, $D_{\mathcal{G}}(F)$ is a free* divisor for*

$$\mathrm{Derlog}^*(D_{\mathcal{G}}(F)) \quad = \quad \textit{module (Lie algebra) of } \mathcal{G}\textit{–liftable vector fields.}$$

ii) If instead $\mathcal{G}$ has a Cohen–Macaulay reduction $\mathcal{G}^*$. Then, $D_{\mathcal{G}}(F)$ is a free divisor for*

$$\mathrm{Derlog}^*(D_{\mathcal{G}}(F)) \quad = \quad \textit{module (Lie algebra) of } \mathcal{G}^*\textit{–liftable vector fields.}$$

Remark . We observe that despite the notation, $\mathcal{G}$ and $\mathcal{F}$ typically denote a collection of groups and spaces which depend upon n, p, and (for unfoldings) q. Whether $\mathcal{G}$ or $\mathcal{G}^*$ are Cohen–Macaulay or whether there is genericity of $\mathcal{G}$ or $\mathcal{G}^*$–liftable vector fields may depend upon the dimensions n and p, so results may include conditions on dimensions.

It follows that despite the failure of genericity of Morse type singularities, we may still conclude for discriminants that the module of liftable vector fields defines a free* divisor structure. As an example of this we consider the consequence for $\mathcal{K}_V$ equivalence for nonlinear sections $f_0 : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$ of a free divisor $V, 0$. Even if $n < hn(V)$ (the holonomic codimension of V), frequently V does not generically have Morse–type singularities in dimension n. Nonetheless, provided $n < hn(V)$, the discussion in Example 2.5 implies $\mathcal{K}_V$ is Cohen–Macaulay. Hence, Theorem 2 allows us to conclude.

Theorem 3. *Suppose that $V, 0 \subset \mathbb{C}^p, 0$ is a free divisor and that F is a $\mathcal{K}_V$–versal unfolding of $f_0 : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$ with $n < hn(V)$. Then the $\mathcal{K}_V$–discriminant $D_V(F)$ is a free* divisor for*

$$\mathrm{Derlog}^*(D_V(F)) \quad = \quad \textit{module (Lie algebra) of } \mathcal{K}_V\textit{–liftable vector fields.}$$

As applications of this we give several corollaries.

For finitely $\mathcal{A}$–determined germs $f_0 : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$, we proved in part I that if f_0 belongs to the "distinguished bifurcation class" of germs then the bifurcation set of the $\mathcal{A}$–versal unfolding is a free divisor. This restriction excludes a number of classes of germs occurring in the nice dimensions (in the sense of Mather [M]). We can extend this to all of the nice dimensions as follows.

Corollary 2.9. *Suppose that $f_0 : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$ is a finitely $\mathcal{A}$– determined germ with (n, p) in the nice dimensions. Then, the bifurcation set $B(f)$ for the $\mathcal{A}$–versal unfolding f of f_0 is a free* divisor for*

$$\mathrm{Derlog}^*(B(f)) \quad = \quad \textit{the module of } \mathcal{A}\textit{–liftable vector fields.}$$

In fact, by the argument for proving Theorem 3 of Part I, let f_0 be obtained from f by pull-back by a germ g_0. Then, the bifurcation set $B(f) = \mathcal{K}_{D(f)}$–discriminant for g, the $\mathcal{K}_{D(f)}$–versal unfolding of g_0. By [D2] and [D6] the $\mathcal{K}_{B(f)}$–liftable vector fields are the $\mathcal{A}$–liftable vector fields. That (n, p) is in the nice dimensions then implies that $n < hn(D(f))$ so that Theorem 1 applies.

Second, we consider the case of the special $\mathcal{A}$–equivalence of complete intersection map germs f_0 on isolated complete intersections $X, 0$, where we allow both f_0 and X to vary. In [MM], Mond and Montaldi show that deformations of such germs can be analyzed by the equivalence of sections of the discriminant of the complete intersection $X_0 = f_0^{-1}(0)$. For a limited range of dimensions given by [D6, Theorem 9.3], the bifurcation set for the versal unfolding will be a free divisor. More generally, we can relax the dimension restriction.

Corollary 2.10. *Suppose that $f_0 : X, 0 \to \mathbb{C}^p, 0$ is a complete intersection map germ defining an isolated singularity $X_0, 0$ on an isolated complete intersection singularity $X, 0$. Let $F : \mathcal{X} \to \mathbb{C}^{p+q}$ be the versal deformation for this special $\mathcal{A}$ equivalence (induced by the versal unfolding of the germ defining X_0). If $X_0, 0$ is an ICIS defined by a simple germ, then the bifurcation set $B(F)$ is a free* divisor for*

$$\mathrm{Derlog}^*(B(F)) \quad = \quad \textit{the module of } \mathcal{K}_{D(F)}\textit{–liftable vector fields.}$$

We can still use the argument of Mond–Montaldi [MM] to represent f_0 as a pullback of the versal unfolding F of the germ defining X_0 by a section of $D(F)$. Then, $D(F)$ is free by [L] and the simplicity of the germ defining X_0 implies that $hn(D(F)) = \infty$. Thus, Theorem 1 applies.

The third consequence is for nonlinear arrangements. Only Boolean arrangements (arrangements consisting of all coordinate hyperplanes) generically have Morse–type singularities; nonetheless for general free hyperplane arrangements we can conclude the following.

Corollary 2.11. *Suppose that $A \subset \mathbb{C}^p$ is a free hyperplane arrangement. Let $A_0 = f_0^{-1}(A)$ be a (nonlinear) arrangement defined via $f_0 : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$. Then, the $\mathcal{K}_A$–discriminant of the $\mathcal{K}_A$–versal unfolding of f_0 is a free* divisor for*

$$\mathrm{Derlog}^*(D_A(F)) \quad = \quad \textit{the module of } \mathcal{K}_A\textit{–liftable vector fields.}$$

We now turn to the proofs of Theorems 1 and 2, although we prove them in the reverse order.

Proof of Theorem 2. The proof we give is a modification of an argument of Looijenga [L] but we use instead Saito's criterion [Sa] for a free divisor. We give the proof in the case that $\mathcal{G}$ has a Cohen–Macaulay reduction $\mathcal{G}^*$. For the case that $\mathcal{G}$ itself is Cohen–Macaulay we just apply the argument given with $\mathcal{G}^* = \mathcal{G}$.

We suppose $f_0 : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$ has finite $\mathcal{G}$–codimension and that n and p are dimensions for which $\mathcal{G}$ has a Cohen–Macaulay reduction $\mathcal{G}^*$. Then, by assumption, f_0 has finite $\mathcal{G}^*$–codimension. Suppose $F \in \mathcal{F}_{un}(q)$ is an unfolding of f_0 and has the form $F(x,u) = (\bar{F}(x,u), u)$ with

$$\varphi_i \quad = \quad \partial_i F \quad = \quad \frac{\partial \bar{F}}{\partial u_i}_{|u=0}, \qquad 1 \leq i \leq q. \tag{2.7}$$

If $\{\varphi_1, \dots, \varphi_q\}$ spans $N\mathcal{G}_e^* \cdot f_0$, then since $T\mathcal{G}_e^* \cdot f_0 \subseteq T\mathcal{G}_e \cdot f_0$, it follows by the versality theorem (see [D5]) that F is both a $\mathcal{G}^*$ and $\mathcal{G}$–versal unfolding of f_0. Moreover, suppose $dim_{\mathbb{C}} N\mathcal{G}_e \cdot f_0 = q'$ and that F_1 is a restriction of F which is a $\mathcal{G}$–versal unfolding on q' parameters. Then, by the same versality theorem, F is a $\mathcal{G}$–trivial extension of F_1 on $q - q'$ parameters. Hence, $D_{\mathcal{G}^*}(F) = D_{\mathcal{G}}(F) \simeq D_{\mathcal{G}}(F_1) \times \mathbb{C}^{q-q'}$. Hence, since a free* divisor structure is defined stably for some suspension $D_{\mathcal{G}}(F_1) \times \mathbb{C}^{\ell}$, it is enough to consider F. If we take $\varphi_q = 0$, then $\dfrac{\partial \bar{F}}{\partial u_q} = 0$, and $\dfrac{\partial}{\partial u_q}$ will be a $\mathcal{G}^*$–liftable vector field and (by the standard construction) an Euler–like vector field.

Next, we consider the $\mathcal{G}^*_{un,e}$ normal space

$$N\mathcal{G}^*_{un,e} \cdot F \quad = \quad T\mathcal{F}_{un,e} / T\mathcal{G}^*_{un,e} \cdot F \tag{2.8}$$

It follows from the Preparation theorem (for adequate systems of DA algebras [D5, §6]) that $N\mathcal{G}^*_{un,e} \cdot F$ is a finitely generated $\mathcal{O}_{\mathbb{C}^q,0}$–module on the generators $\{\varphi_1, \dots, \varphi_q\}$. Hence, we have an exact sequence

$$0 \longrightarrow \mathcal{L} \xrightarrow{\ \beta\ } \mathcal{O}_{\mathbb{C}^q,0}\{\frac{\partial}{\partial u_i}\} \xrightarrow{\ \alpha\ } N\mathcal{G}^*_{un,e} \cdot F \longrightarrow 0 \tag{2.9}$$

where the map α sends $\dfrac{\partial}{\partial u_i} \mapsto \varphi_i$ and $\mathcal{L}$ denotes the kernel of α. Then, $\mathcal{L}$ will be our Derlog$^*(D_{\mathcal{G}}(F))$.

We first characterize $\mathcal{L}$ as the module of "$\mathcal{G}^*$–liftable vector fields". As

$$d\alpha_F(-\xi) \in T\mathcal{G}^*_{un,e} \cdot F \qquad \text{and} \qquad d\alpha_F(\eta) = \alpha(\eta)$$

(2.1) is equivalent to $\eta \in \mathcal{L}$. Thus, at least $\mathcal{L}$ is the module of $\mathcal{G}^*$–liftable vector fields. Next we summarize the key properties of $\mathcal{L}$ by the following proposition

Proposition 2.12. *Let F be the $\mathcal{G}^*$ versal unfolding of f_0 given in (2.7). For $\mathcal{L}$ given in the exact sequence (2.9):*

1. *$\mathcal{L}$ is the Lie algebra of $\mathcal{G}^*$–liftable vector fields;*
2. *$\mathcal{L}$ contains an Euler–like vector field;*
3. *$\mathcal{L}$ is a free $\mathcal{O}_{\mathbb{C}^q,0}$–module;*
4. *$\mathcal{L}$ is of rank q so β is given by a $q \times q$–matrix, whose determinant (which is the generator for the 0–th Fitting ideal of $N\mathcal{G}^*_{un,e} \cdot F$) defines the $\mathcal{G}^*$–discriminant of F; hence, $supp(\theta_q/\mathcal{L}) = D_{\mathcal{G}}(F)$.*

Proof of Proposition 2.12. We have already seen that $\mathcal{L}$ is the set of $\mathcal{G}^*$–liftable vector fields and contains an Euler–like vector field. Next, to see that $\mathcal{L}$ is a Lie algebra, we observe that it is the tangent space to the group of $\mathcal{G}^*$–liftable diffeomorphisms on $\mathbb{C}^q, 0$, as described in remarks preceding Definition 2.2. By a standard argument, $\mathcal{L}$ is a Lie algebra.

To establish 3), we observe by 1) of the definition of CM–reduction that $N\mathcal{G}^*_{un,e} \cdot F$ is Cohen–Macaulay as an $\mathcal{O}_{\mathbb{C}^q,0}$–module. Hence, it is a Cohen–Macaulay module over $\mathcal{O}_D$, where

$$D \quad = \quad \mathrm{supp}(N\mathcal{G}^*_{un,e} \cdot F) \quad (= D_{\mathcal{G}}(F)).$$

Since $\dim D = q-1$, the Auslander–Buchsbaum formula implies that $N\mathcal{G}^*_{un,e} \cdot F$ has projective dimension 1. Thus, it follows that in (2.9) that $\mathcal{L}$ is a free $\mathcal{O}_{\mathbb{C}^q,0}$–module.

Finally, it remains to prove 4). By 3), we know that $\mathcal{L}$ is a free $\mathcal{O}_{\mathbb{C}^q,0}$–module. As β is an inclusion, its rank is no greater than q, however, it cannot be less than q as $\mathrm{supp}(N\mathcal{G}^*_{un,e} \cdot F)$ has dimension $q-1$. Thus, $\mathcal{L}$ has rank q, and $\mathrm{supp}(N\mathcal{G}^*_{un,e} \cdot F)$ is defined by the vanishing of $\det(\beta)$ (which is the generator of the 0–th Fitting ideal of $N\mathcal{G}^*_{un,e} \cdot F$). This completes 4) and the proof of Proposition 2.12. □

To finish the proof of Theorem 2, we note that by 3) of the definition of CM–reduction,

$$\begin{aligned} \mathrm{supp}(N\mathcal{G}^*_{un,e} \cdot F) &= \mathrm{supp}(N\mathcal{G}_{un,e} \cdot F) \\ &= D_{\mathcal{G}}(F). \end{aligned} \tag{2.10}$$

Thus, $\mathrm{supp}(\theta_q/\mathcal{L}) = D_{\mathcal{G}}(F)$.

Finally, since $\mathcal{G}$ has geometrically defined discriminants, $\mathcal{L} \subseteq \mathrm{Derlog}(D_{\mathcal{G}}(F))$.

□

Proof of Theorem 1. We use the notation in the proof of Theorem 2. Again we give the proof when $\mathcal{G}$ has a CM–reduction $\mathcal{G}^*$.

If we extend the $\mathcal{G}$–versal unfolding F_1 to a $\mathcal{G}^*$–versal unfolding F, then

$$D_{\mathcal{G}^*}(F) \quad = \quad D_{\mathcal{G}}(F) \quad \simeq \quad D_{\mathcal{G}}(F_1) \times \mathbb{C}^{q-q'}. \tag{2.11}$$

Thus, it is sufficient to prove that $D_{\mathcal{G}}(F)$ is a free divisor.

The module of $\mathcal{G}^*$–liftable vector fields is the free $\mathcal{O}_{\mathbb{C}^q,0}$–module on q generators $\mathrm{Derlog}^*(D_{\mathcal{G}}(F)) \subseteq \mathrm{Derlog}(D_{\mathcal{G}}(F))$. We apply Saito's criterion to a set of free generators of $\mathrm{Derlog}^*(D_{\mathcal{G}}(F)) = \mathcal{L}$ given by Proposition 2.12. Thus, it is sufficient to show $\det(\beta)$, which is a generator for the 0–th Fitting ideal, is a reduced defining equation for $D_{\mathcal{G}}(F)$.

We consider the Zariski open subset Z of $D_{\mathcal{G}}(F)_{reg}$ given in Definition 2.8. At a point $u \in Z$, we may choose local coordinates $u_1 = h, u_2, \dots, u_q$. Then, by (2) of Definition 2.8, there are $\zeta_i \in \mathrm{Derlog}^*(D_{\mathcal{G}}(F))$ such that $\zeta_1 = u_1 \cdot \dfrac{\partial}{\partial u_1}$ and $\zeta_{i(u)} = \dfrac{\partial}{\partial u_i}, i = 2, \dots, q$. Hence, the determinant of the coefficients of the ζ_i has the form unit $\cdot\, u_1$ which defines $D_{\mathcal{G}}(F)$ near u. Thus, the 0–th Fitting ideal defines $D_{\mathcal{G}}(F)$ with reduced structure near u. Hence, $\det(\beta)$ defines $D_{\mathcal{G}}(F)$ with reduced structure. By Saito's criterion, $D_{\mathcal{G}}(F)$ is a free divisor with $\mathrm{Derlog}(D_{\mathcal{G}}(F))$ generated by a set of generators of $\mathrm{Derlog}^*(D_{\mathcal{G}}(F))$, so they agree.

□

3. $\mathcal{K}_V$–Discriminants of Complete Intersections

In this section, we consider $\mathcal{K}_V$–Discriminants of versal unfoldings for sections of nonisolated complete intersections $V, 0$. We first identify a special class of "free complete intersection singularities" extending the class of free divisors.

Suppose the complete intersection $V, 0 \subset \mathbb{C}^p, 0$ is defined by the equation $H : \mathbb{C}^p, 0 \to \mathbb{C}^k, 0$. Just as for divisors, we define

$$\mathrm{Derlog}(H) \quad = \quad \{\zeta \in \theta_p : \zeta(H) = 0\}.$$

This is the module of vector fields tangent to the level sets of H. The associated sheaf *Derlog*(H) is coherent.

Definition 3.1. A complete intersection $V, 0 \subset \mathbb{C}^p$ defined by $H : \mathbb{C}^p, 0 \to \mathbb{C}^k, 0$ ($p \geq k$) will be called an *(H–) free complete intersection* if $\mathrm{Derlog}(H)$ is a free $\mathcal{O}_{\mathbb{C}^p,0}$–module. Then, H will be called a *free defining equation* for V.

Remark . The coherence of *Derlog*(H) implies that if V is H–free, then $\mathrm{Derlog}(H)$ will have rank $p - k$ as an $\mathcal{O}_{\mathbb{C}^p,0}$ –module.

We will frequently abuse terminology and call V a free complete intersection. For example, a product of free divisors is a free complete intersection (see §5 for properties and examples). For a free divisor a good defining equation is a free defining equation (but not conversely, see [D3, Example 2.11 d]).

In general, the properties of free complete intersections follow those of free divisors very closely. Nonlinear sections have a singular Milnor fiber and one can compute the singular Milnor number and higher multiplicities as for nonlinear sections of free divisors (see [D3, Parts 2, 3]). One might then expect that just as for isolated singularities and free divisors, the $\mathcal{K}_V$–discriminants of versal deformations of nonlinear sections of a free complete intersection $V, 0$ will also be free, provided V generically has Morse–type singularities in the appropriate sense. We next consider a simple example for which this is not the case. This example was worked out with Anne Frühbis–Krüger, using a package she developed for the computer algebra system "Singular"[Sg]. We make use of results from §§5 and 7.

Example 3.2. We consider the free complete intersection $V = V_1 \times V_2 \subset \mathbb{C}^5$, where $V_1 = \{(x, y, z) \in \mathbb{C}^3 : h_1(x, y, z) = xyz = 0\}$ and $V_2 = \{(w, v) \in \mathbb{C}^2 : h_2(w, v) = wv = 0\}$ are both Boolean arrangements, and hence free divisors. Also, by Lemma 7.7 of part I, each V_i generically has Morse type singularities in all dimensions. Hence, by Proposition 7.5 below, the product $V = V_1 \times V_2$ also genericially has Morse–type singularities in all dimensions.

Next, we define a linear section $f_0 : \mathbb{C}^2, 0 \to \mathbb{C}^5, 0$ by $f_0(s, t) = (s + 2t, s + t, t - s, s - 3t, t - 5s)$. By choosing the weights of all variables to equal one, f_0 and V are both weighted homogeneous. By Proposition 5.6 below, $\mathrm{Derlog}(V)$ is generated by:

$$\{x\frac{\partial}{\partial x}, \quad y\frac{\partial}{\partial y}, \quad z\frac{\partial}{\partial z}, \quad w\frac{\partial}{\partial w}, \quad v\frac{\partial}{\partial v}, \quad h_2\frac{\partial}{\partial x}, \quad h_2\frac{\partial}{\partial y}, \quad h_2\frac{\partial}{\partial z}, \quad h_1\frac{\partial}{\partial w}, \quad h_1\frac{\partial}{\partial v}\}$$

Using these we can compute $N\mathcal{K}_V \cdot f_0$ and verify by the infinitesimal criterion for versality for $\mathcal{K}_V$–versality (Theorem 1 of [D1]) that a $\mathcal{K}_V$–versal unfolding $F(s, t, u) = (\bar{F}(s, t, u), u)$ of f_0 where $u = (u_1, \ldots, u_5)$ is given by

$$\bar{F}(s, t, u) = f_0(s, t) + (0, 0, u_1, u_2 t + u_3, u_4 t + u_5).$$

For F to be weighted homogeneous, we must assign weights $\mathrm{wt}(u_1, u_2, u_3, u_4, u_5) = (1, 0, 1, 0, 1)$. Thus, although the module of $\mathcal{K}_V$–liftable vector fields is graded, the graded pieces are not finite dimensional, but are modules over the ring of germs on (u_2, u_4). However, the package of Anne Frühbis–Krüger, "KVequiv.lib"for the program Singular, enables one to find generators of fixed weight and low degree in (u_2, u_4). From among low weight $\mathcal{K}_V$–liftable vector fields, were identified $\zeta_i, i = 0, \dots, 5$ of weights $0, 1, 1, 2, 2, 3$, such that their (u_2, u_4) components after reducing mod (u_2, u_4) were respectively: $(0, 0)$, $(u_1, *)$, $(u_5, *)$, $(u_3^2, *)$, $(0, u_5^2 + g_1)$, and $(0, u_3^3 + g_2)$. Here ζ_0 is the Euler vector field and has u_5 component equal to u_5; and g_1 does not contain as a nonzero term u_3^2. Also, there is no element with (u_2, u_4) component $(0, u_5)$; nor do the terms obtained from ζ_1, ζ_2, and ζ_3 with u_4 component 0 contain terms with pure powers u_5^2 or u_3^3 in the u_2 components. Thus, it follows that all 6 of the ζ_i are needed as generators so the number of generators ≥ 6, and the module of $\mathcal{K}_V$–liftable vector fields is not a free module.

Thus, for sections of free complete intersections, the best one can hope for in general is that discriminants are free* divisors (see the discussion in §8).

We shall establish this by giving a Cohen–Macaulay reduction $\mathcal{K}_V^*$ for $\mathcal{K}_V$ and applying Theorem 2. To define $\mathcal{K}_V^*$, we recall the subgroup $\mathcal{K}_H$ of $\mathcal{K}_V$ which preserves the level sets of H (see e.g. [DM, §3]). The subgroup of diffeomorphisms $\mathcal{K}_V^*$ lies between $\mathcal{K}_H$ and $\mathcal{K}_V$. It consists of elements of $\mathcal{K}_V$ (i.e. diffeomorphisms of $\mathbb{C}^p \times \mathbb{C}^n, 0$) which when restricted to $V \times \mathbb{C}^n$ are induced by restrictions of diffeomorphisms from $\mathcal{K}_H$ (see §6).

We must replace the holonomic codimension $hn(V)$, used for free divisors, by $h(V)$. This is the codimension of the set of y for which $T_{log}(h)_{(y)} \neq T_y S_i$, where S_i the canonical Whitney stratum of V containing y. For example, by an argument in [D3, §7], for the free complete intersection $V = \prod_{i=1}^r V_i$, $h(V) = min\{h(V_i)\} + r - 1$.

Theorem 4. *Let $V, 0 \subset \mathbb{C}^p, 0$ be a free complete intersection of codimension k. If $k \leq n < h(V)$, then for nonlinear sections $f_0 : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$, $\mathcal{K}_V^*$ is a Cohen–Macaulay reduction for $\mathcal{K}_V$. Hence, for a $\mathcal{K}_V$–versal unfolding F of f_0, the $\mathcal{K}_V$–discriminant $D_V(F)$ is a free* divisor for*

$$\mathrm{Derlog}^*(D_V(F)) \quad = \quad \text{module (Lie algebra) of } \mathcal{K}_V^*\text{–liftable vector fields.}$$

In light of Theorem 1, we ask when we have the genericity of $\mathcal{K}_V^*$–liftable vector fields. In fact this almost never happens. It is not due to the failure of genericity of Morse–type singularities for free complete intersections. We shall see in §7 that this does often hold. Rather, it is due to $\mathcal{K}_V$–liftable vector fields which are not $\mathcal{K}_V^*$–liftable. The exception is for the case of smooth V. As a result, we recover the result of Looijenga [L] that for isolated complete intersection singularities (which correspond to $V = \{0\}$) the discriminant $D(F)$ for a versal unfolding F is a free divisor, with $\mathrm{Derlog}(D(F))$ the module of $\mathcal{K}_V$–liftable vector fields.

4. Vanishing Topology for Sections of Free* Divisors

Given the abundance of free* divisors following from the results in the preceding sections, we now consider the vanishing topology of nonlinear sections of a free* divisor $V, 0$. As $V, 0$ is a hypersurface, nonlinear sections have a singular Milnor fiber and singular Milnor number. We shall give a formula for the singular Milnor number as the length of a determinantal module as in [DM], except that we will find it necessary to subtract certain multiplicities of "virtual singularities".

Let $V, 0 \subset \mathbb{C}^p, 0$ be a free* divisor. We suppose, if necessary, that we have replaced V by an appropriate suspension. We also suppose H is a good defining equation for V with Euler–like vector field e. Let $f_0 : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$ be a nonlinear section of $V, 0$ which has finite $\mathcal{K}_V$– codimension. We recall [DM, §4] that a topological stabilization of f_0 consists of a family $f_t : U \to \mathbb{C}^p$, where U is neighborhood of 0 in $\mathbb{C}^n$, satisfying : i) when $t = 0$, f_t is a representative of the germ f_0 which is geometrically transverse to V on $U \backslash \{0\}$, and ii) f_t is geometrically transverse to V on U for all sufficiently small $t \neq 0$. Then, by [DM, §4], for a sufficiently small ball B_ϵ about 0 of radius $\epsilon > 0$, $f_t^{-1}(V) \cap B_\epsilon$ is, up to homeomorphism, independent of t and is homotopy equivalent to a bouquet of spheres of dimension $n-1$. This is the singular Milnor fiber, and the number of spheres is the singular Milnor number, denoted $\mu_V(f_0)$.

To compute this number for free* divisors, we note Derlog$^*(V)$ defines a singular, possibly nonintegrable, distribution on V. We must take into account the "virtual singularities"where f_t fails to be algebraically transverse to the distribution.

Definition 4.1. For a stabilization f_t, a *virtual singularity* for a given t is a point $x \in U$ such that $f_t(x) \in V$ but

$$df_t(x)(\mathbb{C}^n) + T_{log^*}(H)_{(f_t(x))} \neq T_{(f_t(x))}\mathbb{C}^p.$$

By [DM], the codimension of the module $T\mathcal{K}_{H,e} \cdot f_0$ computes the singular Milnor number in the case of a free divisor V. We introduce an analogue for the case of a free*-divisor V.

$$T\mathcal{K}^*_{H,e} \cdot f_0 \quad \stackrel{\text{def}}{=} \quad \mathcal{O}_{\mathbb{C}^n,0}\{\frac{\partial f_0}{\partial x_1}, \dots, \frac{\partial f_0}{\partial x_n}, \zeta_1 \circ f_0, \dots, \zeta_{p-1} \circ f_0\}$$

where Derlog$^*(H)$ is generated by $\zeta_1, \dots, \zeta_{p-1}$. The normal space is given by

$$N\mathcal{K}^*_{H,e} \cdot f_0 \quad = \quad \mathcal{O}^p_{\mathbb{C}^n,0}/T\mathcal{K}^*_{H,e} \cdot f_0. \tag{4.1}$$

Then, $\mathcal{K}^*_{H,e}$–codim$(f_0, x) = \dim_{\mathbb{C}} N\mathcal{K}^*_{H,e} \cdot f_0$.

Remark . Although $T\mathcal{K}^*_{H,e} \cdot f_0$ is not always an extended tangent space for a group action, it is in the case that $V = D_{\mathcal{G}}(F)$ is a discriminant with $\mathcal{G}$ Cohen–Macaulay or having CM–reduction $\mathcal{G}^*$. Then, $\mathcal{K}^*_H$ is a subgroup of diffeomorphisms $\mathcal{K}_V$ defined by replacing diffeomorphisms preserving V by $\mathcal{G}$ (resp. $\mathcal{G}^*$) liftable diffeomorphisms preserving the level sets of H (see §6).

For a virtual singularity x of f_t, we let $\mathcal{K}^*_{H,e}$–codim(f_t, x) denote the $\mathcal{K}^*_{H,e}$–codimension of the germ $f_t : \mathbb{C}^n, x \to \mathbb{C}^p, f_t(x)$. Then, we obtain the formula for the singular Milnor number.

Theorem 5. *Suppose $V, 0 \subset \mathbb{C}^p, 0$ is a free* divisor with good defining equation H. Let $f_0 : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$ be a nonlinear section of $V, 0$ which has finite $\mathcal{K}^*_H$–codimension. For f_t a stabilization of f_0,*

$$\mu_V(f_0) \quad = \quad \mathcal{K}^*_{H,e}\text{–}codim(f_0) \; - \; \sum \mathcal{K}^*_{H,e}\text{–}codim(f_t, x_i) \tag{4.2}$$

where the sum is over the finite number of virtual singular points x_i for a given sufficiently small value of t.

For t sufficiently small, all of the virtual singular points will occur in a sufficiently small neighborhood of 0.

Proof of Theorem 5. We outline the argument which follows along the lines of [DM, §5] except using the reduction. The module $N\mathcal{K}^*_{H,e} \cdot f_0$ is a determinantal module.

If we sheafify the corresponding module for the stabilization f_t we obtain a sheaf $\mathcal{N}$ on a neighborhood U of 0 which is the quotient sheaf of $\mathcal{O}_U^p$ by the subsheaf which is an $\mathcal{O}_U$–module generated by

$$\{\frac{\partial f_t}{\partial x_1}, \dots, \frac{\partial f_t}{\partial x_n}, \zeta_1 \circ f_t, \dots, \zeta_{p-1} \circ f_t\}.$$

The support of $\mathcal{N}$ is one–dimensional and is then Cohen–Macaulay by results of Macaulay [Mc] and Northcott [No]. For the projection $\pi : \mathbb{C}^{n+1} \to \mathbb{C}$, its restriction to supp($\mathcal{N}$) is finite to one so $\pi_*(\mathcal{N})$ is Cohen–Macaulay (and in fact free) over $\mathbb{C}$. Thus,

$$\begin{aligned} \dim_{\mathbb{C}} N\mathcal{K}_{H,e}^* \cdot f_0 &= \dim_{\mathbb{C}}(\pi_*(\mathcal{N})/t \cdot \pi_*(\mathcal{N})) \\ &= \dim_{\mathbb{C}}(\pi_*(\mathcal{N})/(t-t_0) \cdot \pi_*(\mathcal{N})) \\ &= \sum \dim_{\mathbb{C}} N\mathcal{K}_{H,e}^* \cdot (f_t, x_i) \end{aligned} \tag{4.3}$$

Here the sum in (4.3) is over points $(x_i, t_0) \in \text{supp}(\mathcal{N})$. This sum splits into two parts, a sum over points in $V \times \{t_0\}$ and a sum over points in the complement. For $(f_{t_0}(x_i), t_0) = (y_i, t_0) \notin V \times \{t_0\}$, by assumption, $T_{log^*}(V)_{(y_i)} = T_{y_i}\mathbb{C}^p$ so the generators $\{\zeta_1, \dots, \zeta_p\}$ evaluated at y_i are linearly independent. Hence, $\{\zeta_1, \dots, \zeta_{p-1}\}$ span the tangent space to the level set of H at $y_i = f_{t_0}(x_i)$. Hence, the argument in Lemma 5.6 of [DM] implies that $\dim_{\mathbb{C}} N\mathcal{K}_{H,e}^* \cdot (f_t, x_i)$ equals the Milnor number of $H \circ f_{t_0}$ at x_i. Using complex Morse theory as in [DM, §4] yields that the sum over $(x_i, t_0) \in \text{supp}(\mathcal{N}) \backslash V$ equals the singular Milnor number of f_0. The remaining terms correspond to points in $\text{supp}(\mathcal{N}) \cap V$ which are exactly the virtual singularities each contributing $\mathcal{K}_{H,e}^*$–codim(f_t, x_i).

□

Ultimately, we would like to reexpress the terms on the RHS of (4.2) in terms of invariants of $N\mathcal{K}_{V,e} \cdot f_0$ (or possibly $N\mathcal{K}_{H,e} \cdot f_0$); however the dimensions of these modules do not behave well under deformation. Another invariant of finite length $\mathcal{O}_{\mathbb{C}^p,0}$–modules N is the Buchsbaum-Rim multiplicity [BRm], which we denote by $m_{BR}(N)$. For determinantal modules such as $N = N\mathcal{K}_{H,e}^* \cdot f_0$, $m_{BR}(N) = \dim_{\mathbb{C}} N$. Hence, we may restate Equation (4.3) for computing the singular Milnor number solely in terms of Buchsbaum-Rim multiplicities

$$\mu_V(f_0) \quad = \quad m_{BR}(N\mathcal{K}_{H,e}^*(f_0)) \; - \; \sum m_{BR}(N\mathcal{K}_{H,e}^*(f_t, x_i)) \tag{4.4}$$

with the sum is over the finite number of virtual singular points x_i for a given sufficiently small value of t. This leads us to the basic question.

Question. Can we replace the terms on the RHS of (4.4) by expressions involving the Buchsbaum-Rim multiplicities of the modules $N\mathcal{K}_{V,e} \cdot f_0$ (or possibly $N\mathcal{K}_{H,e} \cdot f_0$), and allowing certain virtual singularities?

Remark . In the weighted homogeneous case, for a Cohen–Macaulay reduction $\mathcal{G}^*$ of a group $\mathcal{G}$, we give in [D9] a formula for the number of vanishing cycles in a stabilization of an unfolding f of f_0 in terms of the normal space of $\mathcal{G}^*$.

Next, we seek circumstances when we can simplify the expression representing the sum over the virtual singularities. In the first special case, we suppose that for the section f_0, all stabilizations f_t have virtual singularities on a curve $\mathcal{C}$. For each irreducible branch $\mathcal{C}_i$ of $\mathcal{C}$, there is a Zariski open subset of jets of germs of sections $g : \mathbb{C}^n, 0 \to \mathbb{C}^p, y$ with $y \in \mathcal{C}_i$, with $N\mathcal{K}_{(H,y),e}^* \cdot g$ having minimum dimension c_i.

Then, we may perturb any stabilization so that at any virtual singular point x with $f_t(x) = y \in \mathcal{C}_i$ we have $\mathcal{K}^*_{H,e}$–codim$(f_t, x) = c_i$. Then, we can compute the singular Milnor number as follows. Let $(D(f_0), \mathcal{C}_i)$ denote the intersection multiplicity of the discriminant $D(f_0)$ and $\mathcal{C}_i$ (where $D(f_0)$ has nonreduced structure so we count intersection points $y \in D(f_0)$ with multiplicity given by the number of critical points mapping to y).

Corollary 6. *Suppose $V, 0 \subset \mathbb{C}^p, 0$ is a free* divisor with good defining equation H. Let $f_0 : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$ be a nonlinear section of $V, 0$ of finite $\mathcal{K}^*_H$–codimension with the property that any stabilization of f_0 has its virtual singularities on the curve $\mathcal{C}$. Then,*

$$\mu_V(f_0) \quad = \quad \mathcal{K}^*_{H,e}\text{-}codim(f_0) \; - \; \sum_i c_i \cdot (D(f_0), \mathcal{C}_i) \tag{4.5}$$

Proof. We may first perturb f_0 by an isotopy so that f_t is transverse (i.e. $D(f_t)$ is transverse) to $\mathcal{C}$. If f_t is not $1-1$ on its critical set, we further perturb it so it does become so while still remaining transverse to $\mathcal{C}$. Then, on each branch $\mathcal{C}_i$ there will be $(D(f_0), \mathcal{C}_i)$ transverse intersection points. The number of such transverse points will remain fixed under an additional sufficiently small perturbation. We may further slightly perturb f_t if necessary to a stabilization f_t which only has virtual singularities x with $f_t(x) \in \mathcal{C}_i$ of $\mathcal{K}^*_{H,e}$–codim$(f_t, x) = c_i$. Hence, the sum on the RHS of (4.3) becomes exactly the sum on the right hand side of (4.5). □

Example 4.2 ("Twisted Whitney Umbrellas")**.** As a first example, consider a section of the Whitney umbrella V in (1.6) with the free* divisor structure with Derlog$^*(V)$ (given by D_1 for $(a, b) = (1, 0)$) generated by $\{e = \eta_1, \eta_3, Z\eta_4 - W^2\eta_2\}$. Consider a section of the form $(Y, Z, W) = g_0(X, y, z) = (y, z, p(X, y))$ where $X = (x_1, \dots, x_{n-1})$ so $g_0 : \mathbb{C}^{n+1}, 0 \to \mathbb{C}^3, 0$. If we view V as the image of $F = (y^2, uy, u)$, then the pullback of F by g_0 yields the germ $f_0(X, y) = (y^2, yp(X, y^2), X)$ with $f_0 : \mathbb{C}^n, 0 \to \mathbb{C}^{n+1}, 0$. In the case $n = 1$ this gives a family of germs appearing in Mond's classification [Mo1]. For f_0 to have finite $\mathcal{A}$–codimension, we must have that $(p_X, yp_y) = (p_{x_1}, \dots, p_{x_{n-1}}, yp_y)$ generates an ideal of finite codimension. The singular Milnor number $\mu_V(g_0)$ equals the number of vanishing cycles for the image of a generic perturbation of f_0. In fact, for the case of germs $f_0 : \mathbb{C}^2, 0 \to \mathbb{C}^3, 0$, Mond [Mo2], de Jong–Van Straten [JVS], etc have directly related this to $\mathcal{A}_e$–codim(f_0).

We compute the singular Milnor number for general n. In carrying out computations, to reduce notation, we represent the ring $\mathcal{O}_{\mathbb{C}^{n+1},0}$ by the coordinates for the particular $\mathbb{C}^{n+1}$, e.g. $\mathcal{O}_{X,y,z}$. From

$$\frac{\partial g_0}{\partial x_i} \; = \; (0, 0, p_{x_i}), \quad \frac{\partial g_0}{\partial y} \; = \; (1, 0, p_y), \quad \frac{\partial g_0}{\partial z} \; = \; (0, 1, 0) \tag{4.6}$$

we obtain by projection along $\dfrac{\partial g_0}{\partial y}$ and $\dfrac{\partial g_0}{\partial z}$ onto the third component

$$N\mathcal{K}^*_{H,e} \cdot g_0 \simeq \mathcal{O}_{X,y}/(p_X, p^2(p - 2yp_y))$$

Thus,

$$\dim_{\mathbb{C}}(N\mathcal{K}^*_{H,e} \cdot g_0) \quad = \quad 2\dim_{\mathbb{C}}\mathcal{O}_{X,y}/(p_X, p) + \dim_{\mathbb{C}}(\mathcal{O}_{X,y}/(p_X, p - 2yp_y) \tag{4.7}$$

A straightforward calculation shows that the virtual singularities only occur along the "handle "of the Whitney umbrella, i.e. the Y–axis; and that a generic virtual singularity has $\mathcal{K}^*_{H,e}$ − codim = 2. The intersection number of $D(g_0)$ with the Y–axis is given by the number of points where $z = p = p_X = 0$. Hence, it equals $\dim \mathcal{O}_{X,y}/(p_X, p)$. Thus, by Corollary 6 and (4.7)

$$\mu_V(g_0) \quad = \quad \dim_{\mathbb{C}} \mathcal{O}_{X,y}/(p_X, p - 2yp_y)$$

If p is weighted homogeneous, this is exactly $\mathcal{A}_e - \text{codim}(g_0)$, see [Mo1] for the case of $n = 2$.

Example 4.3. The second type of example we consider is a weighted homogeneous surface singularity which is a free* divisor by Proposition 1.7. Let $V, 0 \subset \mathbb{C}^3, 0$ be a weighted homogeneous surface singularity and let $V' = V \times \mathbb{C}$, with last coordinate w. Given a nonlinear section of V, $f_1 : \mathbb{C}^n, 0 \to \mathbb{C}^3, 0$ we construct a section of V', $f : \mathbb{C}^n, 0 \to \mathbb{C}^4, 0$ defined using coordinates $x = (x_1, \ldots, x_n)$ for $\mathbb{C}^n$ and $y = (y_1, y_2, y_3)$ by $f(x) = (y, w) = (f_1(x), f_2(x))$. If f_{1t} denotes a stabilization of f_1 as a section of V, then $f_t = (f_{1t}, f_{2t})$ is a stabilization of f as a section of V'. Moreover, their singular Milnor fibers are homeomorphic. It suffices to consider f and f_t and $N\mathcal{K}^*_{He}$.

As an example, consider a Pham–Brieskorn singularity $V, 0 \subset \mathbb{C}^3, 0$ defined by $h(y) = y_1^p + y_2^m + y_3^\ell = 0$, and let $\text{wt}(y_i) = a_i$. We consider an "A_{k-1}"type section of $V \times \mathbb{C}$

$$f(x) \quad = \quad (f_1(x), f_2(x)) \quad = \quad (x_1^k + \sum_{i=4}^{n} x_i^2, x_2, x_3, x_1),$$

and consider a deformation where we only deform f_1.

To apply Theorem 5, we recall from the proof of Proposition 1.7 the determinantal vector fields $\eta_{ij} = \frac{\partial h}{\partial y_i}\frac{\partial}{\partial y_j} - \frac{\partial h}{\partial y_j}\frac{\partial}{\partial y_i}$, and the generators for $\text{Derlog}^*(h)$ given by

$$(4.8) \qquad \zeta_i \;=\; \eta_{jk} + a_i y_i \frac{\partial}{\partial w} \quad i = 1, 2, 3 \qquad \text{and} \qquad \zeta_0 \;=\; -de \;=\; -\sum a_i y_i \frac{\partial}{\partial y_i}.$$

where for $i > 0$, (ijk) is a cyclic permutation of (123).

$$\frac{\partial f}{\partial x_1} = (\frac{\partial f_1}{\partial x_1}, 0, 0, 1), \quad \frac{\partial f}{\partial x_2} = (0, 1, 0, 0) \quad \frac{\partial f}{\partial x_3} = (0, 0, 1, 0)$$
$$\text{and} \qquad \frac{\partial f}{\partial x_j} = (2x_j, 0, 0, 0) \quad \text{for } j \geq 4.$$

If we project $N\mathcal{K}^*_{He} \cdot f$ off the free submodule generated by $f_{x_i}, i = 1, 2, 3$ onto the first component and take the quotient by $f_{x_i}, i \geq 4$, we obtain from the generators in (4.8)

$$(4.9) \;\; N\mathcal{K}^*_{He} \cdot f \quad \simeq \quad \mathcal{O}_{\mathbb{C}^3}/(-a_1 g_1 \frac{\partial g_1}{\partial x_1}, \frac{\partial h}{\partial y_3} \circ g - a_2 x_2 \frac{\partial g_1}{\partial x_1}, -\frac{\partial h}{\partial y_2} \circ g - a_3 x_3 \frac{\partial g_1}{\partial x_1})$$

with $g : \mathbb{C}^3, 0 \to \mathbb{C}^4, 0$ obtained from f by setting $x_i = 0, i \geq 4$. Considering degrees in (4.9) we obtain $\dim_{\mathbb{C}} N\mathcal{K}^*_{He} \cdot f = (2k-1)(\ell-1)(m-1)$.

Second, to determine the contribution from virtual singularities, we have $x_i = 0$ for $i \geq 4$, $h = 0$, and the generators in (4.9), with g replaced by g_t, vanish. From the first generator, either $g_{1\,t}$, or $\frac{\partial g_{1\,t}}{\partial x_1} = 0$. If $\frac{\partial g_{1\,t}}{\partial x_1} = 0$, setting the second and third generators equal to 0 implies that $y_2 = y_3 = 0$. We only consider points where $h \circ f_t = 0$; thus $y_1 = 0$ so $g_{1\,t} = 0$. However, generically we can assume both are not 0, so we may assume $\frac{\partial g_{1\,t}}{\partial x_1} \neq 0$. Then, there are k solutions to $g_{1\,t} = 0$ and $(\ell - 1)(m - 1)$ solutions to the second and third generators $= 0$. At each such virtual singular point, they generate the maximal ideal so the corresponding local algebra as in (4.9) has dimension 1. Hence, Theorem 5 gives

$$\begin{aligned} \mu_V(f_1) = \mu_{V'}(f) &= (2k-1)(\ell-1)(m-1) - k(\ell-1)(m-1) \\ &= (k-1)(\ell-1)(m-1). \end{aligned} \tag{4.10}$$

This is also the Milnor number of the Pham–Brieskorn singularity $x_1^k + x_2^m + x_3^\ell + \sum_{i=4}^n x_i^2 = 0$. However, the singular Milnor fiber for a stabilization such as $(x_1^k + \sum_{i=4}^n x_i^2 + t)^p + x_2^m + x_3^\ell = 0$ must have singularities along $x_1^k + \sum_{i=4}^n x_i^2 + t = x_2 = x_3 = 0$, and is superficially quite different from the Milnor fiber of this Pham–Brieskorn singularity. Is there an intrinsic way of seeing the relation between these apparently quite distinct objects?

Example 4.4. Corollary 7 also applies to free divisors when $n \geq hn(V)$. As a last example, consider the surface singularity $V, 0 \subset \mathbb{C}^3$ defined by $F(x,y,z) = f(x,y) + zg(x,y)$, where both f and g are weighted homogeneous with f defining an isolated curve singularity and $\mathrm{wt}(g) \geq \mathrm{wt}(f)$. Let J(f) be the Jacobian ideal of f. Suppose $(J(f) : g)$ is a complete intersection ideal with generators $\{h_1, h_2\}$ satisfying the numerical condition

$$\mathrm{wt}(h_1) + \mathrm{wt}(h_2) + \mathrm{wt}(g) \quad = \quad \mathrm{wt}(x) + \mathrm{wt}(y) + \mathrm{wt}(H)$$

where H is the Hessian of F. Then V is a free divisor but with nonholonomic stratum the z–axis [D8]. A generic virtual singularity is given by a Morse–type singularity for $\mathcal{K}$–equivalence (Example 7.7) of the form

$$\varphi(x_1, \dots, x_n) \quad = \quad (x_1, x_2, z_0 + \sum_{j=3}^{n} x_j^2)$$

whose discriminant is transverse to the z–axis. A straightforward computation shows

$$\dim N\mathcal{K}^*_{H,e} \cdot \varphi \quad = \quad \dim_{\mathbb{C}} \mathcal{O}_{x,y}/(J(f) : g),$$

the colength of the ideal $(J(f) : g)$. Thus, the singular Milnor number for a nonlinear section $\varphi : \mathbb{C}^n, 0 \to \mathbb{C}^3, 0$ is given by

$$\mu_V(\varphi) = \mathcal{K}^*_{H,e}\text{-codim}(\varphi) \;-\; \mathrm{col}(J(f) : g) \cdot (D(\varphi), z\text{-axis}). \tag{4.11}$$

II: Discriminants of Sections of Free Complete Intersections

5. Free Complete Intersections

We establish in this section several basic properties of free complete intersections. Suppose V is a free complete intersection with free defining equation h, we first remark that $V \times \mathbb{C}^q$ is a free complete intersection with free defining equation $h \circ \pi$ where $\pi : \mathbb{C}^{p+q} \to \mathbb{C}^p$ denotes projection. This follows from the following simple Lemma whose proof is straightforward.

Lemma 5.1. *Let $u = (u_1, \dots, u_q)$ denote local coordinates for $\mathbb{C}^q$, and let $\zeta_i \in \theta_p, 1 \leq i \leq r$. Then,*

$$\mathrm{Derlog}(h) \quad = \quad \mathcal{O}_{\mathbb{C}^p,0}\{\zeta_1, \dots, \zeta_r\}$$

iff

$$\mathrm{Derlog}(h \circ \pi) \quad = \quad \mathcal{O}_{\mathbb{C}^{p+q},0}\{\zeta_1, \dots, \zeta_r, \frac{\partial}{\partial u_1}, \dots, \frac{\partial}{\partial u_q}\}.$$

Next, we give a simple criterion for establishing the freeness of complete intersections.

Proposition 5.2. *Suppose $V_i, 0 \subset \mathbb{C}^p, 0, i = 1, \dots, r$ are divisors (not necessarily irreducible) with reduced defining equations $h_i : \mathbb{C}^p, 0 \to \mathbb{C}, 0$ such that:*

(1) *$V' = \cup V_i, 0 \subset \mathbb{C}^p, 0$ is a free divisor (defined by $\prod h_i$);*
(2) *there exist Euler–like vector fields e_i for h_i such that $e_i(h_j) = \delta_{ij} h_j$;*
(3) *$V = \cap V_i, 0$ is a complete intersection defined by $h = (h_1, \dots, h_r)$.*

Then, $V, 0$ is a free complete intersection with free defining equation h.

Proof. We observe that $\mathrm{Derlog}(h) \subset \mathrm{Derlog}(V')$, and $\mathrm{Derlog}(V')$ is a free $\mathcal{O}_{\mathbb{C}^p,0}$–module. It is sufficient to prove $\mathrm{Derlog}(h)$ is a direct summand of $\mathrm{Derlog}(V')$ for then it is a finitely generated projective $\mathcal{O}_{\mathbb{C}^p,0}$–module, hence free. We define a projection

$$\psi : \mathrm{Derlog}(V') \to \mathrm{Derlog}(h)$$

$$\xi \mapsto \xi - \sum_{i=1}^{r} \frac{\xi(h_i)}{h_i} e_i. \tag{5.1}$$

To see it is well–defined, let $\xi \in \mathrm{Derlog}(V')$. Then, ξ is tangent to $V_{i\,reg} \backslash \cup_{j \neq i} V_j$. Thus, $\xi(h_i)$ vanishes on $V_{i\,reg} \backslash \cup_{j \neq i} V_j$, and hence its closure V_i. Hence, $\xi(h_i) = \varphi_i h_i$ and $\frac{\xi(h_i)}{h_i} = \varphi_i$. At least $\psi(\xi) \in \theta_p$.

Moreover, it is easily checked that ψ is $\mathcal{O}_{\mathbb{C}^p,0}$–linear. It remains to check that ψ is projection onto $\mathrm{Derlog}(h)$. However, it is straightforward to check that: if $\xi \in \mathrm{Derlog}(V')$, then $\psi(\xi)(h_j) = 0$ all j, and if $\xi \in \mathrm{Derlog}(h)$, then $\psi(\xi) = \xi$. This completes the proof. □

We list several consequences.

Corollary 5.3. *The product $V, 0 = \prod V_i, 0 \subset \mathbb{C}^p$ of free divisors $V_i, 0 \subset \mathbb{C}^{p_i}, 0, i = 1, \dots, r$ with good defining equations h_i is a free complete intersection with free defining equation $h = (h_1 \circ \pi_1, \dots, h_r \circ \pi_r)$, where $\pi_j : \mathbb{C}^p \to \mathbb{C}^{p_j}$ denotes projection.*

Proof. Let $V_i' = V_i \times \prod_{j \neq i} \mathbb{C}^{p_j}$, which is still free with defining equation $h_i \circ \pi_i$. Then, $V' = \cup V_i'$ is the production union of the V_i and hence also free. Also, the Euler–like vector fields e_i for each h_i can be trivially extended to $\mathbb{C}^p$ and satisfy 2) of Proposition 5.2. Finally, $V (= \prod V_i) = \cap V_i'$. Thus, by Proposition 5.2, $V, 0 = \prod V_i, 0$ is a free complete intersection with free defining equation $h = (h_1 \circ \pi_1, \dots, h_r \circ \pi_r)$. □

As a consequence of Corollary 5.3, we deduce that products of any of the free divisors listed in Part I, such as discriminants of versal unfoldings, Coxeter arrangements, etc. yield free complete intersections. These are the main examples. The simplest such is the product of $\{0\} \subset \mathbb{C}$ yielding the free complete intersection $\{0\} \subset \mathbb{C}^p$. The product of r discriminants of versal unfoldings yields a free complete intersection which is part of another discriminant of a versal unfolding where at least r singular multigerms appear.

Remark . Products of free divisors form a very special class of nonisolated complete intersection singularities. However, a nonlinear section of such a product is the intersection of the pullbacks of the individual free divisors. For a nonlinear section algebraically transverse in a punctured neighborhood, such pullbacks are "almost free complete intersections"[D3] which naturally extend the class of ICIS.

A second consequence is a weaker form of Proposition 5.2.

Corollary 5.4. *Suppose $V_i, 0 \subset \mathbb{C}^p, 0, i = 1, \dots, r$ are divisors with reduced defining equations $h_i : \mathbb{C}^p, 0 \to \mathbb{C}, 0$ such that:*

(1) *$V' = \cup V_i, 0 \subset \mathbb{C}^p, 0$ is a free divisor (defined by $\prod h_i$);*
(2) *$V = \cap V_i, 0$ is a complete intersection defined by $h = (h_1, \dots, h_r)$.*

Let $\mathbb{C}^r$ have coordinates $(t_1, \dots, t_r)$. Then, $V \times \mathbb{C}^r, 0$ is a free complete intersection with free defining equation $h' = (e^{t_1} h_1 \circ \pi, \dots, e^{t_r} h_r \circ \pi)$, where $\pi : \mathbb{C}^{p+r} \to \mathbb{C}^p$ denotes projection.

Proof. We replace each V_i by $V_i' = V_i \times \mathbb{C}^r$, and replace h_i by $h_i' = e^{t_i} h_i \circ \pi$. Then, $e_i = \dfrac{\partial}{\partial t_i}$ is an Euler–like vector field for h_i' and $e_i(h_j) = 0$ if $i \neq j$. Then, $V'' = V' \times \mathbb{C}^r = \cup V_i'$ is still a free divisor (defined by $\prod h_i'$), and $V \times \mathbb{C}^r = \cap V_i'$. Thus, proposition 5.2 applies, yielding the result.

□

That products of free divisors are free complete intersections extends to products of free complete intersections.

Corollary 5.5. *The product of free complete intersections $V_i, 0 \subset \mathbb{C}^{p_i}, 0, i = 1, \dots, r$ with free defining equations h_i is free with free defining equation $h = (h_1 \circ \pi_1, \dots, h_r \circ \pi_r)$, where again $\pi_i : \mathbb{C}^p \to \mathbb{C}^{p_i}$ denotes projection.*

Proof. The proof is by induction on r and reduces to verifying the result for $r = 2$. Denote $h = (h_1 \circ \pi_1, h_2 \circ \pi_1)$, and let $x = (x_1, \dots, x_{p_1})$ and $y = (y_1, \dots, y_{p_2})$ denote local coordinates for $\mathbb{C}^{p_1}$ and $\mathbb{C}^{p_2}$. Also, let

$$\text{Derlog}(h_i) = \mathcal{O}_{\mathbb{C}^{p_i},0}\{\zeta_1^{(i)}, \dots, \zeta_{r_i}^{(i)}\} \tag{5.2}$$

where $r_i = p_i - k_i$. Finally let $p = p_1 + p_2$. Then,

$$\text{Derlog}(h) \quad = \quad \text{Derlog}(h_1 \circ \pi_1) \cap \text{Derlog}(h_2 \circ \pi_2) \tag{5.3}$$

(still $\pi_i : \mathbb{C}^p \to \mathbb{C}^{p_i}$ denotes projection). By Lemma 5.1

$$\text{Derlog}(h_1 \circ \pi_1) = \mathcal{O}_{\mathbb{C}^p,0}\{\zeta_1^{(1)}, \dots, \zeta_{r_1}^{(1)}\} \oplus \mathcal{O}_{\mathbb{C}^p,0}\{\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_{p_2}}\} \tag{5.4}$$

with an analogous formula for $\text{Derlog}(h_2 \circ \pi_2)$. Each summand respects the decomposition

$$\theta_p \quad = \quad \mathcal{O}_{\mathbb{C}^p,0}\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_{p_1}}\} \oplus \mathcal{O}_{\mathbb{C}^p,0}\{\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_{p_2}}\}.$$

Hence, by (5.3) and (5.4)

$$\text{Derlog}(h) \quad = \quad \mathcal{O}_{\mathbb{C}^p,0}\{\zeta_1^{(1)}, \dots, \zeta_{r_1}^{(1)}\} \oplus \mathcal{O}_{\mathbb{C}^p,0}\{\zeta_1^{(2)}, \dots, \zeta_{r_2}^{(2)}\}. \tag{5.5}$$

□

By contrast, the module structure $\text{Derlog}(V_1 \times V_2)$ becomes increasingly complicated. We use the above notation, except now the V_i are free divisors so $r_i = p_i$.

Proposition 5.6. *Suppose $V_i, 0 \subset \mathbb{C}^{p_i}$ are free divisors with good defining equations $h_i : \mathbb{C}^{p_i}, 0 \to \mathbb{C}, 0$ for $i = 1, 2$. Let $p = p_1 + p_2$. Then*

$$\begin{aligned} \mathrm{Derlog}(V_1 \times V_2) = \mathcal{O}_{\mathbb{C}^p,0}\{\zeta_i^{(1)},\ 1 \le i \le p_1, \quad \zeta_i^{(2)},\ 1 \le i \le p_2, \\ h_2\frac{\partial}{\partial x_i},\ 1 \le i \le p_1, \quad h_1\frac{\partial}{\partial y_i},\ 1 \le i \le p_2\} \end{aligned} \tag{5.6}$$

In particular, even if we restrict to $V = V_1 \times V_2$ which is a complete intersection of dimension $p - 2$, we obtain

$$\mathrm{Derlog}(V)_{|V} = \mathcal{O}_{V,0}\{\zeta_i^{(1)},\ 1 \le i \le p_1, \quad \zeta_i^{(2)},\ 1 \le i \le p_2\} \subset \mathcal{O}_{V,0}^{(p)} \tag{5.7}$$

and $\mathrm{Derlog}(V)_{|V}$ has p generators which will fail to give it good algebraic properties for either discriminants or for computing the vanishing topology of nonlinear sections. This observation motivates the introduction of Cohen–Macaulay reduction for free complete intersections.

Proof of Proposition 5.6. We have Euler–like vector fields e_i for h_i so that $e_i(h_j) = \delta_{ij}h_j$. Let M denote the module on the RHS of (5.6). We easily see $M \subseteq \mathrm{Derlog}(V)$.

For the reverse inclusion, let $\zeta \in \mathrm{Derlog}(V)$. We may write $\zeta = \zeta_1 + \zeta_2$ with $\zeta_i \in \theta(\pi_i)$, for π_i the projection onto $\mathbb{C}^{p_i}$. Hence, $\zeta(h_i) = \zeta_i(h_i)$. We claim $\zeta_i \in M, i = 1, 2$.

For example, we may write $\zeta(h_1)(= \zeta_1(h_1)) = \alpha_1 h_1 + \alpha_2 h_2$. Then,

$$(\zeta_1 - \alpha_1 e_1)(h_1) = (\zeta - \alpha_1 e_1)(h_1) = \alpha_2 h_2.$$

As $\zeta_1 - \alpha_1 e_1 \in \theta(\pi_1)$, $\alpha_2 h_2 \in (\frac{\partial h_1}{\partial x_1}, \dots, \frac{\partial h_1}{\partial x_{p_1}})$, the ideal in $\mathcal{O}_{\mathbb{C}^p,0}$ generated by the $\frac{\partial h_1}{\partial x_i}$. However, $\frac{\partial h_1}{\partial x_i} \in \mathcal{O}_{\mathbb{C}^{p_1},0}$ and $h_2 \in \mathcal{O}_{\mathbb{C}^{p_2},0}$. Hence, h_2 is not a zero divisor in $\mathcal{O}_{\mathbb{C}^p,0}/(\frac{\partial h_1}{\partial x_1}, \dots, \frac{\partial h_1}{\partial x_{p_1}})$. Thus, $\alpha_2 \in (\frac{\partial h_1}{\partial x_1}, \dots, \frac{\partial h_1}{\partial x_{p_1}})$. If $\alpha_2 = \sum b_i \frac{\partial h_1}{\partial x_i}$, let

$$\xi \quad = \quad \zeta_1 - \alpha_1 e_1 - \sum b_i h_2 \frac{\partial}{\partial x_i}.$$

Then $\xi(h_1) = 0$ and $\xi \in \theta(\pi_1)$. Hence, by Lemma 5.1, $\xi = \sum \lambda_j \zeta_j^{(1)}$. Thus,

$$\zeta_1 \quad = \quad \alpha_1 e_1 + \sum b_i h_2 \frac{\partial}{\partial x_i} + \sum \lambda_j \zeta_j^{(1)},$$

showing $\zeta_1 \in M$. □

Remark . The conclusion of Proposition 5.6 extends in a straightforward fashion to finite products.

6. CM–reduction for $\mathcal{K}_V$–Equivalence for Free Complete Intersections

Before we begin the proof of Theorem 3, we give more detailed information about the group $\mathcal{K}_V^*$. We recall that it is the subgroup of diffeomorphisms in $\mathcal{K}_V$ which when restricted to $V \times \mathbb{C}^n$ are induced by restrictions of diffeomorphisms from $\mathcal{K}_H$. Here $\mathcal{K}_H$–equivalence is defined by the subgroup of $\mathcal{K}_V$ given by

$$\mathcal{K}_H \quad = \quad \{\Psi \in \mathcal{K} : \tilde{H} \circ \Psi = \tilde{H}\}.$$

for $\tilde{H}$ the composition of H with projection onto $\mathbb{C}^p$. The extended tangent space is given by

$$T\mathcal{K}_{H,e} \cdot f_0 \;=\; \mathcal{O}_{\mathbb{C}^n,0}\{\frac{\partial f_0}{\partial x_1}, \dots, \frac{\partial f_0}{\partial x_n}, \zeta_1 \circ f_0, \dots, \zeta_r \circ f_0\} \tag{6.1}$$

where $\operatorname{Derlog}(H)$ is generated by $\zeta_1, \dots, \zeta_r$.

We have the inclusions

$$\mathcal{K}_H \quad \subset \quad \mathcal{K}_V^* \quad \subset \quad \mathcal{K}_V .$$

It is easily checked that $\mathcal{K}_V^*$ is a geometrically defined subgroup of $\mathcal{A}$ or $\mathcal{K}$. To give the extended tangent spaces, we note that $I(V) \cdot \theta_p \subset \operatorname{Derlog}(V)$ and

$$T\mathcal{K}_{V,e}^* \cdot f_0 \;=\; \mathcal{O}_{\mathbb{C}^n,0}\{\frac{\partial f_0}{\partial x_1}, \dots, \frac{\partial f_0}{\partial x_n}, \zeta_1 \circ f_0, \dots, \zeta_r \circ f_0\} + I(V) \cdot \theta(f_0) \tag{6.2}$$

where again $\zeta_1, \dots, \zeta_r$ generate $\operatorname{Derlog}(H)$. If V is a free complete intersection, then $r = p - k$.

In addition, consider an unfolding $F : \mathbb{C}^{n+q}, 0 \to \mathbb{C}^{p+q}, 0$ of f_0. We write $F(x, u) = (\bar{F}(x, u), u)$ where we use local coordinates u for $\mathbb{C}^q$. There are defined the associated tangent spaces for the unfolding groups. Let $\{\zeta_1, \dots, \zeta_m\}$ be a set of generators for $\operatorname{Derlog}(V)$. Then, the extended tangent spaces for the unfolding groups are given by

$$T\mathcal{K}_{V,un,e} \cdot F \;=\; \mathcal{O}_{\mathbb{C}^{n+q},0}\{\frac{\partial \bar{F}}{\partial x_1}, \dots, \frac{\partial \bar{F}}{\partial x_n}, \zeta_1 \circ \bar{F}, \dots, \zeta_m \circ \bar{F}\}$$

and

$$T\mathcal{K}_{V,un,e}^* \cdot F \;=\; \mathcal{O}_{\mathbb{C}^{n+q},0}\{\frac{\partial \bar{F}}{\partial x_1}, \dots, \frac{\partial \bar{F}}{\partial x_n}, \zeta_1 \circ \bar{F}, \dots, \zeta_r \circ \bar{F}\} + I(V) \cdot \theta(\bar{F}). \tag{6.3}$$

We note that the normal space for $\mathcal{K}_V^*$ can be written

$$\begin{aligned} N\mathcal{K}_{V,un,e}^* \cdot F \;&=\; \theta(\bar{F}) / T\mathcal{K}_{V,un,e}^* \cdot F \\ &=\; \mathcal{O}_{\mathcal{X},0}\{\frac{\partial}{\partial y_i}\} / \mathcal{O}_{\mathcal{X},0}\{\frac{\partial \bar{F}}{\partial x_1} \dots, \frac{\partial \bar{F}}{\partial x_n}, \zeta_1 \circ \bar{F}, \dots, \zeta_r \circ \bar{F}\} \end{aligned} \tag{6.4}$$

where $\mathcal{X} = \bar{F}^{-1}(V)$.

To begin the proof of Theorem 3 that $\mathcal{K}_V^*$ is a CM–reduction of $\mathcal{K}_V$, we first establish condition (2) of 2.6. The codimensions are related by the next proposition.

Proposition 6.1. *Suppose $V, 0 \subset \mathbb{C}^p, 0$ is a free complete intersection with free defining equation $H : \mathbb{C}^p, 0 \to \mathbb{C}^k, 0$. If $n < h(V)$, then, $f_0 : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$ has finite $\mathcal{K}_V$–codimension iff it has finite $\mathcal{K}_V^*$–codimension.*

Proof. Let $\mathcal{TK}_{V,e} \cdot f_0$, $\mathcal{TK}_{V,e}^* \cdot f_0$, $\mathcal{NK}_{V,e} \cdot f_0$, and $\mathcal{NK}_{V,e}^* \cdot f_0$ denote the associated sheaves to the tangent and normal spaces $T\mathcal{K}_{V,e} \cdot f_0$, $T\mathcal{K}_{V,e}^* \cdot f_0$, $N\mathcal{K}_{V,e} \cdot f_0$, and $N\mathcal{K}_{V,e}^* \cdot f_0$. Then, by the nullstellensatz for coherent analytic sheaves, finite $\mathcal{K}_V$–codimension is equivalent to $\operatorname{supp}(\mathcal{NK}_V \cdot f_0) = \{0\}$, and similarly for $\mathcal{K}_V^*$–equivalence.

We claim that the condition $n < hn(V)$ implies that

$$\operatorname{supp}(\mathcal{NK}_{V,e} \cdot f_0) \quad = \quad \operatorname{supp}(\mathcal{NK}_{V,e}^* \cdot f_0). \tag{6.5}$$

In fact, this follows from a more general result for unfoldings which we now consider.

□

Associated to each of these normal spaces for an unfolding F of f_0, we have sheaves denoted by $\mathcal{NK}_{V,un,e} \cdot F$, and $\mathcal{NK}^*_{V,un,e} \cdot F$ which are sheaves of $\mathcal{O}_{\mathbb{C}^{n+q}}$–modules. The relation between the supports of the normal sheaves is given by the following.

Proposition 6.2. *Suppose $V, 0 \subset \mathbb{C}^p, 0$ is a free complete intersection with free defining equation $H : \mathbb{C}^p, 0 \to \mathbb{C}^r, 0$, with $n < h(V)$. Let F be an unfolding of $f_0 : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$. Then, as sheaves of $\mathcal{O}_{\mathbb{C}^{n+q}}$–modules,*

$$supp(\mathcal{NK}_{V,un,e} \cdot F) \quad = \quad supp(\mathcal{NK}^*_{V,un,e} \cdot F). \tag{6.6}$$

Proof of Proposition 6.2. For the assertion we note that

$$\mathcal{TK}^*_{V,un,e} \cdot F \;\subseteq\; \mathcal{TK}_{V,un,e} \cdot F,$$

so we have the inclusion $\subseteq$ in (6.6). For the reverse inclusion, we examine where they differ. Because of the sheaf inclusion

$$\mathcal{I}(V) \cdot \Theta(\bar{F}) \;\subset\; \mathcal{TK}^*_{V,un,e} \cdot F,$$

a point $(x, u) \in \operatorname{supp}(\mathcal{NK}^*_{V,un,e} \cdot F)$ implies $\bar{F}(x, u) \in V$. If $\bar{F}(x, u) \in V$, then $(x, u) \notin \operatorname{supp}(\mathcal{NK}^*_{V,un,e} \cdot F)$ is equivalent to

$$T_{log}H_{(x)} + D\bar{F}(x, u)(T_{(x,u)}\mathbb{C}^n) \quad = \quad T_{\bar{F}(x,u)}\mathbb{C}^p. \tag{6.7}$$

However, the assumption $n < h(V)$ implies that

$$T_{log}H_{(x)} \quad = \quad T_{log}V_{(x)} \quad = \quad T_xS_i,$$

where S_i is a stratum of the canonical Whitney stratification containing x. Thus, (6.7) implies that $\bar{F}$ is algebraically transverse to V at (x, u) (see [D3, §2]). Thus, $(x, u) \in \operatorname{supp}(\mathcal{NK}^*_{V,un,e} \cdot F)$ implies $(x, u) \in \operatorname{supp}(\mathcal{NK}_{V,un,e} \cdot F)$. Thus, we have equality in (6.6). ☐

Finally, applying Proposition 6.2 to f_0 viewed as an unfolding on 0 parameters, we obtain (6.5), completing the proof of proposition 6.1.

Now we turn to considering the $\mathcal{K}_V$-critical set and discriminant. These were defined in Part I in the case that $V, 0$ was a divisor. However, the definition for $V, 0$ a complete intersection is the same. Briefly, given $V, 0 \subset \mathbb{C}^p, 0$ and $f_0 : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$ which has finite $\mathcal{K}_V$–codimension, let $F : \mathbb{C}^{n+q}, 0 \to \mathbb{C}^{p+q}, 0$ be a $\mathcal{K}_V$–versal unfolding of f_0. We write $F(x, u) = (\bar{F}(x, u), u)$ where we use local coordinates u for $\mathbb{C}^q$. Also, we denote the projection $\pi : \mathbb{C}^{n+q}, 0 \to \mathbb{C}^q, 0$.

Definition 6.3. We define the $\mathcal{K}_V$–*critical set* of F to be

$$C_V(F) \quad = \quad \operatorname{supp}(\mathcal{NK}_{V,un,e} \cdot F).$$

Analogously we define the $\mathcal{K}^*_V$–*critical set* of F to be $C^*_V(F) = \operatorname{supp}(\mathcal{NK}^*_{V,un,e} \cdot F)$

Remark 6.4. It follows from the definition of $C_V(F)$, that it consists of points (x_0, u_0), with $y_0 = \bar{F}(x_0, u_0)$, such that the germ $\bar{F}(\cdot, u_0) : \mathbb{C}^n, x_0 \to \mathbb{C}^p, y_0$ is not algebraicially transverse to V at x_0. As $n < hn(V)$, this is equivalent to the restriction of the projection $\pi : \mathbb{C}^{n+q}, 0 \to \mathbb{C}^q, 0$ to $\mathcal{X}\,(= \bar{F}^{-1}(V))$ not being the germ of a submersion at (x_0, u_0).

Suppose $f_0 : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$ has finite $\mathcal{K}_V$–codimension and $n < h(V)$. By propositions 6.5 and 6.7, if F is an unfolding of the finite $\mathcal{K}_V$–codimension germ f_0, the critical sets $C_V(F)$ for $\mathcal{K}_V$ and $C^*_V(F)$ for $\mathcal{K}^*_V$ agree. Also, $\pi|C_V(F)$ is finite to one. By Grauert's theorem the direct image of sheaves $\mathcal{NK}_V \cdot F \overset{\text{def}}{=} \pi_*(\mathcal{NK}_{V,un,e} \cdot F)$

and $\mathcal{N}\mathcal{K}_V^* \cdot F \stackrel{\text{def}}{=} \pi_*(\mathcal{N}\mathcal{K}_{V,un,e}^* \cdot F)$ are coherent. However, these sheaves have support exactly the $\mathcal{K}_V$ and $\mathcal{K}_V^*$–discriminants of F. Hence, the $\mathcal{K}_V$ discriminant $D_V(F)) = \pi(C_V(F))$ is an analytic subset of the same dimension as $C_V(F)$ and similarly for $D_V^*(F)) = \pi(C_V^*(F))$. Moreover, by proposition 6.7 their images under the projection π agree, so

$$D_V(F)) \quad = \quad \pi(C_V(F)) \quad = \quad \pi(C_V^*(F)) \quad = \quad D_V^*(F).$$

Thus, condition (3) of Definition 2.6 is satisfied. It remains to show that $\mathcal{K}_V^*$ is Cohen–Macaulay, i.e. that $\mathcal{N}\mathcal{K}_V^* \cdot F$ is Cohen–Macaulay with support of dimension $q-1$.

Proposition 6.5. *Suppose that $V, 0 \subset \mathbb{C}^p, 0$ is a free complete intersection of codimension k, with $k \leq n < h(V)$. Then, $\mathcal{N}\mathcal{K}_V^* \cdot F$ is Cohen–Macaulay with supp$(\mathcal{N}\mathcal{K}_V^* \cdot F) = D_V^*(F)$ of dimension $q-1$.*

Proof. Let $f_0 : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$ have finite $\mathcal{K}_V$–codimension and let $F : \mathbb{C}^{n+q}, 0 \to \mathbb{C}^{p+q}, 0$ be a $\mathcal{K}_V$–versal unfolding of f_0. As F is a $\mathcal{K}_V$–versal unfolding of f_0, $\bar{F} : \mathbb{C}^{n+q}, 0 \to \mathbb{C}^p, 0$,viewed as a map, is algebraically transverse to V at 0. Thus, If we write $V = V_0 \times \mathbb{C}^s$ with $T_{log}V_0 = \{0\}$, then $\mathcal{X} = \bar{F}^{-1}(V)$ is diffeomorphic to $V_0 \times \mathbb{C}^{s'}$ (and so is also a complete intersection of dimension $n + q - k$ where k is codimension of V). In particular, it is Cohen–Macaulay.

As V is a free complete intersection with free defining equation H, Derlog(H) is freely generated by say $\zeta_1, \dots, \zeta_{p-k}$. By (6.4) $\mathcal{N}\mathcal{K}_{V,un,e}^* \cdot F$ is an $\mathcal{O}_\mathcal{X}$–module which is the quotient of $\mathcal{O}_\mathcal{X}^p$ by the $\mathcal{O}_\mathcal{X}$–submodule generated by the $n+p-k$ generators

$$\{\frac{\partial \bar{F}}{\partial x_1}(x,u), \dots, \frac{\partial \bar{F}}{\partial x_n}(x,u), \zeta_1 \circ \bar{F}(x,u), \dots, \zeta_{p-k} \circ \bar{F}(x,u)\}.$$

It has support $C_V^*(F)$. Since $n \geq k$, $n + p - k \geq p$, we apply results of Eagon–Northcott [EN] extending those of Macaulay–Northcott [Mc] [No], to conclude that $C_V^*(F)$ is a subvariety of $\mathcal{X}$ of codimension $\leq n + p - k - (p-1) = n - k + 1$ in $\mathcal{X}$ so it has dimension $\geq n + q - k - (n - k + 1) = q - 1$. Moreover, if the codimension is exactly $n - k + 1$ (i.e. dimension exactly $q-1$), then $\mathcal{N}\mathcal{K}_{V,un,e}^* \cdot F$ is a Cohen–Macaulay $\mathcal{O}_\mathcal{X}$–module and $C_V^*(F)$ is Cohen–Macaulay.

If $\dim C_V^*(F) = q$, then so would $\dim \pi(C_V(F)) = \dim \pi(C_V^*(F)) = q$, implying $\pi(C_V(F))$ is locally onto near 0. However, we can now use the definition of $C_V(F)$ as supp$(\mathcal{N}\mathcal{K}_{V,un,e})$. By the parametrized transversality theorem, there are points $u_0 \in \mathbb{C}^q$ arbitrarily close to 0 for which $\bar{F}(\cdot, u_0)$ is geometrically transverse to V. As $n < (h(V) <)hn(V)$, geometric and algebraic transversality agree, see [D2, §3]); thus, $D_V(F) = \pi(C_V(F))$ can not have dimension q. Thus, the dimension is at most $q-1$.

By the above, $\dim C_V^*(F) = \dim C_V(F) \leq q - 1$ so it is exactly $q-1$. Thus, the critical set $C_V^*(F)$ is Cohen–Macaulay of dimension $q-1$ and $\mathcal{N}\mathcal{K}_{V,un,e}^* \cdot F$ is a Cohen–Macaulay $\mathcal{O}_\mathcal{X}$–module. As $\pi | C_V(F)$ is finite to one, we conclude (e.g. using [Se]) that the direct image under a finite map $\mathcal{N}\mathcal{K}_V^* \cdot F = \pi_*(\mathcal{N}\mathcal{K}_{V,un,e}^* \cdot F)$ is Cohen–Macaulay as a $\mathcal{O}_{D_V^*(F)}$–module and hence as a $\mathcal{O}_{\mathbb{C}^q}$–module. Its support is $D_V^*(F) = \pi(C_V^*(F))$ which is, hence, Cohen–Macaulay of dimension $q-1$. This completes the proof of Theorem 4. □

Remark 6.6. We observe that the only two properties of F which were used in proposition 6.10 were: that $\bar{F} : \mathbb{C}^{n+q}, 0 \to \mathbb{C}^p, 0$, viewed as a map, is algebraically transverse to V, and that generically $\bar{F}(\cdot, u_0)$ is tranverse to V. Thus, for any unfolding F with these properties, both $C^*_V(F)$ and $D^*_V(F)$ will be Cohen–Macaulay of dimension $q-1$. Hence, this is true for $C_V(F)$ and $D_V(F)$ using the induced structure.

7. Morse–type Singularities for Sections of General Varieties

We extend the notion of Morse–type singularity defined for sections of divisors to arbitrary analytic germs $V, 0$. We also will see that specific properties which hold for Morse–type singularities for divisors continue to hold for free complete intersections.

Definition 7.1. Given $V, 0 \subset \mathbb{C}^p, 0$ and an integer $n > 0$, a germ $g : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$ is a *Morse–type singularity in dimension* n if g is $\mathcal{K}_V$–equivalent to a germ f_0 which has $\mathcal{K}_{V,e}$–codim $= 1$ and for a common choice of local coordinates, both f_0 and V are weighted homogeneous.

Furthermore, we say V has a *Morse–type singularity in dimension* n at x if there is a germ $g : \mathbb{C}^n, 0 \to \mathbb{C}^p, x$ which is a Morse–type singularity in dimension n (for this, we use $\mathcal{K}_{(V,x)}$–equivalence).

Definition 7.2. V is said to *generically having Morse–type singularities in dimension* n if: all points on the canonical Whitney stata of V of codimension $\leq n+1$ have Morse–type singularities of nonzero exceptional weight type; and any stratum of codimension $> n+1$ lies in the closure of a stratum of codimension $= n+1$.

We first observe that the normal form established in part I for Morse type singularities holds for an arbitrary $V, 0$ not just a free divisor.

Lemma 7.3 (Local Normal Form). *Let $f_0 : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$ be a Morse–type singularity for $V, 0 \subset \mathbb{C}^p, 0$. Then, up to $\mathcal{K}_V$–equivalence, we may assume $V, 0 = \mathbb{C}^r \times V_0, 0$ for $V_0, 0 \subset \mathbb{C}^{p'}, 0$, and with respect to coordinates for which $V_0, 0$ is weighted homogeneous, f_0 has the form*

$$f_0(x_1, \dots, x_n) = (0, \dots, 0, x_1, \dots, x_{p'-1}, \sum_{j=p'}^{n} x_j^2).$$

Proof. The proof given in part I does not specifically refer to V being a free divisor except at one point and there the reference can easily be removed as follows. The initial reduction makes no reference to special properties of V. It allows us to reduce to the case $p' = p$ and show that f_0 can be put in the form

$$f_0(x_1, \dots, x_n) = (x_1, \dots, x_{p-1}, \bar{f}_0(x_1, \dots, x_n)) \tag{7.1}$$

with $d\bar{f}_0(0) = 0$. Then, in (4.15) of [D6], we explicitly give the generators for $T\mathcal{K}_{V,e} \cdot f_0$ and make use of the generators of Derlog(V). Suppose instead that we have a set of generators $\{\zeta_i, i = 0, \dots, \ell\}$ for Derlog(V) with $\ell \geq p$ and with ζ_0 denoting the Euler vector field. By the form of f_0 in (7.1)

$$T\mathcal{K}_{V,e} \cdot f_0 \quad \subseteq \quad \mathcal{O}_{\mathbb{C}^n,0}\{\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_{p-1}}\} \oplus m_n\{\frac{\partial}{\partial y_p}\}. \tag{7.2}$$

As $\mathcal{K}_{V,e} - \text{codim}(f_0) = 1$, we have equality in (7.2). Thus,

$$\frac{\partial}{\partial y_i} \quad \in \quad T\mathcal{K}_{V,e} \cdot f_0 \qquad \text{for } i = 1, \dots, p-1.$$

Also, the projection of $T\mathcal{K}_{V,e} \cdot f_0$ onto $\mathcal{O}_{\mathbb{C}^n,0}\{\frac{\partial}{\partial y_p}\} \simeq \mathcal{O}_{\mathbb{C}^n,0}$ is $m_n\{\frac{\partial}{\partial y_p}\}$. The Euler relation and (7.1) imply for the Euler vector field ζ_0

$$\zeta_0(y_p) \circ f_0 \quad = \quad b_p \bar{f}_0 \; \in \; m_n^2$$

where $b_p = \mathrm{wt}(y_p)$. Hence,

$$\{\zeta_i(y_p) \circ f_0, i = 1, \ldots, \ell; \frac{\partial f_0}{\partial x_j}, j = p, \ldots, n\} \quad \text{generate } m_n. \tag{7.3}$$

Also, by (7.1), $\zeta_i \in m_p \theta_p$ for $i = 1, \ldots, \ell$. Thus,

$$\zeta_i(y_p) \circ f_0 \; \in \; m_{p-1}\mathcal{O}_{\mathbb{C}^n,0} \bmod (m_n^2) \tag{7.4}$$

and, by (7.3), must generate this ideal. Thus,

$$\{\zeta_i(y_p), i = 1, \ldots, \ell\} \quad \text{generate} \quad m_{p-1}\mathcal{O}_{\mathbb{C}^p,0} \bmod ((y_p) + m_p^2). \tag{7.5}$$

After this point, the number of vector fields in $\mathrm{Derlog}(V)$ is no longer of significance and the rest of the proof remains the same. □

Having established the normal form, the arguments given in part I allow us to deduce Corollaries 4.20, 4.22, and 4.24 of [D6] for Morse–type singularities for any V. These three results are summarized as follows.

Corollary 7.4.

(1) *If $V, 0 \subset \mathbb{C}^p, 0$ has a Morse type singularity in dimension n, then it has Morse–type singularities in the "allowable dimensions"$m \geq p' - 1$ where $p' = p - r$ and $r = \dim(T_{log}(V)_0)$.*

(2) *Suppose $V, 0 = \mathbb{C}^r \times V_0, 0$ is weighted homogeneous with $V_0, 0 \subset \mathbb{C}^{p'}, 0$ and $(T_{log}(V_0)_{(0)}) = (0)$. Then V has Morse type singularities in all allowable dimensions iff there is a weighted hyperplane in $\mathbb{C}^{p'}$ which is transverse to the orbits of $Aut_1(V_0)$ in a punctured neighborhood of 0. Here $Aut_1(V_0)$ denotes the group of linearized automorphisms of V_0.*

(3) *If $V, 0 \subset \mathbb{C}^p, 0$ has Morse type singularities, then there is a Zariski open dense subset of Σ_1 consisting of jets of Morse–type singularities (here Σ_1 denotes the 2–jets of germs which are not algebraically transverse to V at 0).*

Also, just as in the hypersurface case, we say that V with Morse type singularities, has "exceptional weight type"positive, negative, or zero type if V satisfies (7.5) with -wt(y_p) having the corresponding positive or negative sign or $= 0$. Then, the next result ensures that having Morse–type singularities is preserved under products.

Proposition 7.5. *If $V_i, 0 \subset \mathbb{C}^{p_i}, 0$ for $i = 1, 2$ have Morse type singularities of the same non–zero exceptional weight type, then the product $V = V_1 \times V_2$ has Morse type singularities of the same exceptional weight type.*

Proof. The proof we give for products is actually very close to the proof of proposition 4.29 given in [D6] for an analogous statement for product–unions.

We let $p = p_1 + p_2$. We may independently make weighted homogeneous change of coordinates to factor $V_i, 0 = \mathbb{C}^{r_i} \times V_{i0}, 0$. Then, the product factors

$$V_1 \times V_2 \quad = \quad \mathbb{C}^{r_1 + r_2} \times (V_{10} \times V_{20}).$$

Hence, we may suppose that $T_{log}V_{i(0)} = (0)$. Furthermore, we suppose that $\mathbb{C}^{p_i - 1}$ are the weighted subspaces given by 2) of corollary 7.4. Then, as each V_i has the same exceptional weight type, we may multiply weights if necessary so that $\mathrm{wt}(y^{(1)}_{p_1}) = \mathrm{wt}(y^{(2)}_{p_2})$ where each V_i has coordinates $(y^{(j)}_i)$. Then, we consider the weighted homogeneous subspace

$$M \quad = \quad (\mathbb{C}^{p_1 -1} \times \{0\}) + (\{0\} \times \mathbb{C}^{p_2 - 1}) + \langle (\frac{\partial}{\partial y^{(1)}_{p_1}}, \frac{\partial}{\partial y^{(2)}_{p_2}}) \rangle.$$

Now, we claim that M satisfies the conditions of 2) of corollary 7.4 so that V has Morse type singularities with the same exceptional weight type. The verification follows exactly the proof of proposition 4.29 of part I, since the vector fields used there which generate $\mathrm{Derlog}(V_1 \uplus V_2)$ also belong to $\mathrm{Derlog}(V)$. □

Remark 7.6. The products of discriminants of versal unfoldings of simple germs or the products of Boolean arrangements are examples of free complete intersections which genericially have Morse–type singularities (of positive exceptional weight) in all dimensions and at any point.

Example 7.7. $\{0\} \subset \mathbb{C}^p$ is a free complete intersection which is a product of the free divisors $\{0\} \subset \mathbb{C}$. It has Morse–type singularities of all dimensions, which are ($\mathcal{K}$ equivalent to) the standard $\Sigma_{n-p+1,0}$ germs

$$f_0(x_1, \dots, x_n) = (x_1, \dots, x_{p-1}, \sum_{j=p}^{n} x_j^2).$$

The final property of Morse type singularities that carry over to general V is the $\mathcal{K}_V$–liftability of $u\dfrac{\partial}{\partial u}$ for the versal unfolding F of a Morse–type singularity in normal form f_0 from Lemma 7.3

$$F(x, u) = (\bar{F}(x, u), u) \qquad \text{with} \qquad \bar{F}(x, u) = f_0(x) + (0, \dots, 0, u). \tag{7.6}$$

Lemma 7.8. *For the $\mathcal{K}_V$–versal unfolding F of a Morse–type singularity in (7.6) with nonzero exceptional weight, the discriminant $D_V(F)$ (defined by $u = 0$) is reduced and $u\dfrac{\partial}{\partial u}$ is a $\mathcal{K}_V$–liftable vector field.*

The proof is exactly as given in Lemma 4.10 of [D6].

8. $\mathcal{K}_V$–Discriminants as Free and Free* Divisors

We conclude by briefly discussing the consequences of our results for $\mathcal{K}_V$–discriminants $D_V(F)$ of versal unfoldings F for free complete intersections $V, 0$. When V is a free divisor which generically has Morse–type singularities, the $\mathcal{K}_V$–discriminants are free divisors. Reexpressed in terms of Theorems 1 and 2, this follows because genericity of Morse–type singularities for V implies the genericity of $\mathcal{K}_V$–liftable vector fields. This together with $\mathcal{K}_V$ being Cohen–Macaulay (for V a free divisor) implies that the module of $\mathcal{K}_V$–liftable vector fields is free and equals $\mathrm{Derlog}(D_V(F)$.

Free complete intersections $V, 0$ still generically have Morse–type singularities in many cases (e.g. by Proposition 7.5). This still implies the genericity of $\mathcal{K}_V$–liftable vector fields by the same arguments given in part I using instead Corollary 7.4 and Lemma 7.8. However, by Example 3.2, we see that the module of $\mathcal{K}_V$–liftable vector fields need not be free.

Suppose instead we use the CM–reduction $\mathcal{K}_V^*$, and V generically has Morse–type singularities (for $\mathcal{K}_V$–equivalence). A sufficient condition for the genericity of $\mathcal{K}_V^*$–liftable vector fields is that for a Morse–type singularity at a point $y \in V$ in a stratum of codimension $\leq n+1$, with the normal form in Lemma 7.3, the vector field $u\dfrac{\partial}{\partial u}$ is $\mathcal{K}_V^*$–liftable. This cannot be true in general by Example 3.2, otherwise by Theorem 1, the $\mathcal{K}_V$–discriminant would be free using the module of $\mathcal{K}_V$–liftable vector fields.

The question reduces to deciding when for a Morse–type singularity in normal form Lemma 7.3, $u\dfrac{\partial}{\partial u}$ is $\mathcal{K}_V^*$–liftable.

We consider the two highest dimensional strata: the smooth stratum, and the strata which up to diffeomorphism have the form $V_1 \times V_2$ where $V_1 \subset \mathbb{C}^{p_1}$ is a free divisor and V_2 is smooth. In the second case, we may assume $V_2 = \mathbb{C}^{p_2-k} \subset \mathbb{C}^{p_2}$. However, $V_2 \times \mathbb{C}^{p_2-k} \subset \mathbb{C}^{p_1+p_2-k}$ is still a free divisor, so we can reduce to the case where $V_2 = \{0\}$. The situation for these cases is given by the next proposition, whose proof is given in part III [D7].

Proposition 8.1. *i) Suppose $V, 0 \subset \mathbb{C}^p, 0$ is smooth. Then, for any unfolding F, $T\mathcal{K}_{V,un,e}^* \cdot F = T\mathcal{K}_{V,un,e} \cdot F$. Hence, any $\mathcal{K}_V$–versal unfolding F is $\mathcal{K}_V^*$–versal, and all $\mathcal{K}_V$–liftable vector fields are also $\mathcal{K}_V^*$–liftable.*

ii) Suppose $V = V_1 \times \{0\} \subset \mathbb{C}^p$, where $V_1, 0 \subset \mathbb{C}^{p_1}, 0$ is a free divisor and $p = p_1 + p_2$ with $p_2 > 0$. Then, for a versal unfolding of a Morse–type singularity in normal form (Lemma 7.3), $u\dfrac{\partial}{\partial u}$ is not $\mathcal{K}_V^*$–liftable.

When $V, 0$ is smooth, the $\mathcal{K}_V$–discriminant is exactly the $\mathcal{K}_V^*$–discriminant by Proposition 6.2. By Proposition 8.1, a Morse–type singularity at any point of V also has $u\dfrac{\partial}{\partial u}$ $\mathcal{K}_{V\,e}^*$–liftable. Thus, by the argument of part I, we have genericity of $\mathcal{K}_{V\,e}^*$–liftable vector fields. Hence, Theorem 1 implies the $\mathcal{K}_V$–discriminant of a versal unfolding is a free divisor, recovering the result of Looijenga.

Corollary 8.2 ((Looijenga [L])). *If V is smooth, the $\mathcal{K}_V$–discriminant for the versal unfolding is a free divisor. In particular, the discriminant for the versal unfolding of an ICIS is a free divisor.*

We note the second part of Proposition 8.1 essentially implies that Example 3.2 is the typical state of affairs; and we will not generally have freeness of $\mathcal{K}_V$–discriminants. Except for ICIS singularities, we must settle for free* divisors.

References

[A] Arnol'd, V. I. *Wave front evolution and equivariant Morse lemma* Comm. Pure App. Math. **29** (1976), 557–582

[A2] ——— *Indices of singular 1-forms on a manifold with boundary, convolution of invariants of reflection groups, and singular projections of smooth surfaces* Russian Math. Surveys **34** (1979), 1–42

[A3] ——— *The Cohomology Ring of the Colored Braid Group* Math. Notes of Acad. Science USSR **5** (1969), 138–140.

[B] Brieskorn, E. *Sur les groupes des tresses (d'après V. I. Arnol'd)* Séminaire Bourbaki,1971/72 Springer Lecture Notes in Math **315** (1973), 21–44.

[Br] Bruce, J.W. *Vector fields on discriminants and bifurcation varieties* London Math. Soc. **17** (1985), 257–262

[BG1] Bruce, J.W. and Giblin, P.J. *Projections of Surfaces with Boundaries* Proc. London Math. Soc. **60** (1990), 392–416

[BG2] ——— *Smooth maps of discriminant varieties* Proc. London Math. Soc. **50** (1985), 535–551

[BRm] Buchsbaum, D. and Rim D. S. *A generalized Koszul Complex II: Depth and Multiplicity* Trans. Amer. Math. Soc. **111** (1964), 197–224

[D1] J. Damon *Deformations of sections of singularities and Gorenstein surface singularities* Amer. J. Math. **109** (1987), 695–722

[D2] ——— *$\mathcal{A}$–equivalence and the equivalence of sections of images and discriminants* Singularity Theory and its Applications: Warwick 1989, Part I Springer Lecture Notes. **1462** (1991), 93-121

[D3] ——— *Higher Multiplicities and Almost Free Divisors and Complete Intersections* Memoirs Amer. Math. Soc. **123 no 589** (1996)

[D4] ——— *Singular Milnor fibers and higher multiplicities for nonisolated complete intersections* Proc. Conf. Sing. and Complex Geom., Ed. Q. Lu et al. AMS/IP Studies in Adv. Math. **5** (1997), 28–53

[D5] ——— *The unfolding and determinacy theorems for subgroups of $\mathcal{A}$ and $\mathcal{K}$* Memoirs Amer. Math. Soc. **50 no. 306** (1984)

[D6] ——— *On the Legacy of Free Divisors: Discriminants and Morse–type Singularities* Amer. J. Math. **120** (1998), 453–492

[D7] ——— *On the Legacy of Free Divisors III: Functions and Divisors on Complete Intersections*, submitted

[D8] ——— *On the Freeness of Equisingular Deformations of Plane Curve Singularities*, Special Issue: Arrangements in Boston, ed. D Cohen and A. Suciu, Topology and Appl. **118** (2002) 31–43

[D9] ———, in preparation

[DM] Damon, J. and Mond, D. *$\mathcal{A}$–codimension and the vanishing topology of discriminants* Invent. Math. **106** (1991), 217–242

[EN] Eagon, J.A. and Northcott, D. *Ideals defined by matrices and a certain complex associated with them* Proc. Roy. Soc. London **299** (1967), 147–172

[Go] Goryunov, V. *Projections of generic surfaces with boundaries* Adv. Soviet Math. **1** (1990), 157–200

[Go2] ——— *Functions on Space Curves* Jour. London Math. Soc. **61** (2000), 807– 822

[Gr] Grandjean, V. *Residual discriminant of a function germ singular along an Euler free divisor* preprint, (2000)

[JVS] de Jong, T. and van Straten , D. *Dimension of the disentanglement component* Singularity Theory and its Applications: Warwick 1989, Part I Springer Lecture Notes. **1462** (1991), 199–211

[L] Looijenga, E.J.N. *Isolated singular points on complete intersections* London Math. Soc. Lecture Notes in Math. **77**, Cambridge Univ. Press (1984)

[Ly] Lyashko, O. *Classification of critical points of functions on a manifold with singular boundary* Funct. Anal. and Appl. **17** (1984), 187–193

[Mc] Macaulay, F.S. *The algebraic theory of modular systems* Cambridge Tracts **19**, Cambridge Univ. Press, 1916

[M] Mather, J *Stability of C^∞–mappings: VI. The Nice Dimensions* Liverpool Singularities Symposium, Springer Lecture Notes **192** (1970), 207-253

[Mo1] Mond, D. *Some remarks on the geometry and classification of germs of maps from surfaces to 3-space* Topology **26** (1987) 361–383

[Mo2] ——— *Vanishing Cycles for Analytic Maps* Singularity Theory and its Applications: Warwick 1989, Part I Springer Lecture Notes. **1462** (1991), 221–234

[Mo3] ——— *personal communication*

[MM] Mond, D. and Montaldi, J. *Deformations of maps on complete intersections, Damon's $\mathcal{K}_V$–equivalence and bifurcations* London Math. Soc. Lect. Notes **201**, Cambridge Univ. Press (1994), 263–284

[MVS] Mond, D. and Van Straten, D. *Milnor number equals Tyurina number for functions on space curves* Jour. London Math. Soc. **63** (2001), 177–187

[No] Northcott, D. G. *Semiregular rings and semi-regular ideals* Quart. J. Math. Oxford **11** (1960), 81-104

[OT] Orlik, P. and Terao, H. *Arrangements of Hyperplanes* Grundlehren der Math. Wiss. **300**, Springer Verlag, 1992

[Sa] Saito, K. *Theory of logarithmic differential forms and logarithmic vector fields* J. Fac. Sci. Univ. Tokyo Sect. Math. **27** (1980), 265–291

[Se] Serre, J. P. *Algèbra Locale Multiplicités* Springer Lecture notes in Math. **11** Springer-Verlag, Berlin., 1965

[Sg] G. M. Greuel, G. Pfister, H. Schonemann,SINGULAR. A computer algebra system for polynomial computations. University of Kaiserlautern (2001),*http://www.singular.uni-kl.de*

[Si] Siersma, D. *Singularities of functions on boundaries, corners, etc.* Quart. J. Math. **32** (1981), 119–127

[Ta] Tari, F. *Projections of Piecewise Smooth Surfaces* J. London Math. Soc. **44** (1991), 155–172

[Te] Teissier, B. *The hunting of invariants in the geometry of the discriminant* in Real and complex singularities, Oslo 1976 (ed. Per Holm), Sijthoof and Noordhoff, Alphen aan den Rijn, 1977, 567–677

[To1] Terao, H. *Arrangements of hyperplanes and their freeness I,II* J. Fac. Sci. Univ. Tokyo Sect. Math. **27** (1980), 293–320

[To2] ——— *The bifurcation set and logarithmic vector fields* Math. Ann. **263** (1983), 313–321

[VS] Van Straten, D. *A note on the discriminant of a space curve* Manuscripta Math. **87** (1995), 167–177

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