CHARACTERISTIC COHOMOLOGY II: MATRIX SINGULARITIES

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ABSTRACT. For a germ of a variety $\mathcal{V}, 0 \subset \mathbb{C}^N, 0$, a singularity $\mathcal{V}_0$ of "type $\mathcal{V}$" is given by a germ $f_0 : \mathbb{C}^n, 0 \to \mathbb{C}^N, 0$ which is transverse to $\mathcal{V}$ in an appropriate sense so that $\mathcal{V}_0 = f_0^{-1}(\mathcal{V})$. In part I of this paper [D6] we introduced for such singularities the Characteristic Cohomology for the Milnor fiber (for $\mathcal{V}$ a hypersurface), and complement and link (for the general case). It captures the cohomology of $\mathcal{V}_0$ inherited from $\mathcal{V}$ and is given by subalgebras of the cohomology for $\mathcal{V}_0$ for the Milnor fiber and complements and is a subgroup for the cohomology of the link. We showed these cohomologies are functorial and invariant under groups of equivalences $\mathcal{K}_H$ for Milnor fibers and $\mathcal{K}_\mathcal{V}$ for complements and links. We also gave geometric criteria for detecting the nonvanishing of the characteristic cohomology.

In this paper we apply these methods in the case $\mathcal{V}$ denotes any of the varieties of singular $m \times m$ complex matrices which may be either general, symmetric or skew-symmetric (with m even). For these varieties we have shown in another paper that their Milnor fibers and complements have compact "model submanifolds" for their homotopy types, which are classical symmetric spaces in the sense of Cartan. As a result, it follows that the characteristic cohomology subalgebras for the Milnor fibers and complements are images of exterior algebras (or in one case a module on two generators over an exterior algebra). In addition, we extend these results to general $m \times p$ complex matrices in the case of the complement and link.

We then apply the geometric detection method introduced in Part I to detect when characteristic cohomology for the Milnor fiber or complement contains a specific exterior subalgebra on ℓ generators and for the link that it contains an appropriate truncated and shifted version of the subalgebra. The detection criterion involves a special type of "kite map germ of size ℓ" based on a given flag of subspaces. The general criterion which detects such nonvanishing characteristic cohomology is then given in terms of the defining germ f_0 containing such a kite map germ of size ℓ. Furthermore we use a restricted form of kite spaces to give a cohomological relation between the cohomology of local links and the global link.

Preliminary Version

INTRODUCTION

Let $\mathcal{V} \subset M$ denote any of the varieties of singular $m \times m$ complex matrices which may be general, symmetric, or skew-symmetric (m even), or $m \times p$ matrices, in the

corresponding space M of such matrices. A "matrix singularity" $\mathcal{V}_0$ of "type $\mathcal{V}$" for any of the $\mathcal{V} \subset M$ is defined as $\mathcal{V}_0 = f_0^{-1}(\mathcal{V})$ by a germ $f_0 : \mathbb{C}^n, 0 \to M, 0$ (which is transverse to $\mathcal{V}$ in an appropriate sense). In part I [D6] we introduced the notion of characteristic cohomology for a singularity $\mathcal{V}_0$ of type $\mathcal{V}$ for any "universal singularity"for the Milnor fiber (in case $\mathcal{V}$ is a hypersurface) and for the complement and link (in the general case). In this paper we determine the characteristic cohomology for matrix singularities in all of these cases.

For matrix singularities the characteristic cohomology will give the analogue of characteristic classes for vector bundles (as e.g. [MS]). For comparison, a vector bundle $E \to X$ over CW complex X is given by map $f_0 : X \to BG$ for G the structure group of E (e.g. O_n, U_n, Sp_n, SO_n, etc.). It is well-defined up to isomorphism by the homotopy class of f_0. Moreover the generators of $H^*(BG; R)$, for appropriate coefficient ring R pull-back via f_0^* to give the characteristic classes of E; so they generate a characteristic subalgebra of $H^*(X; R)$. The nonvanishing of the characteristic classes which then give various properties of E. Various polynomials in the classes correspond to Schubert cycles in the appropriate classifying spaces.

We will give analogous results for categories of matrix singularities of the various types. Homotopy invariance is replaced by invariance under the actions of the groups of diffeomorphisms $\mathcal{K}_H$ or $\mathcal{K}_\mathcal{V}$. For these varieties we have shown in another paper [D3] that they have compact "model submanifolds" for the homotopy types of both the Milnor fibers and the complements and these are classical symmetric spaces in the sense of Cartan. As a result, it will follow that the characteristic subalgebra is the image of an exterior algebra (or in one case a module on two generators over an exterior algebra) on an explicit set of generators.

We give a "detection criterion" for identifying in the characteristic sublgebra an exterior subalgebra on pull-backs of ℓ specific generators of the cohomology of the corresponding symmetric space. It is detected by the defining germ f_0 containing a special type of "unfurled kite map" of size ℓ. This will be valid for the Milnor fiber, complement, and link.

We will do this by using the support of appropriate exterior subalgebras of the Milnor fiber cohomology or of the complement cohomology for the varieties of singular matrices. This is done using results of [D4] giving the Schubert decomposition for the Milnor fiber and the complement to define "vanishing compact models" detecting these subalgebras. In §2 and §6 we use the Schubert decompositions to exhibit vanishing compact models in the Milnor fibers and complements. Then, we use the detection criterion introduced in Part I to give a criterion for detecting nonvanishing exterior subalgebras of the characteristic cohomology using a class of "unfurled kite maps". Matrix singularities $\mathcal{V}_0, 0$ defined by germs f_0 which contain such an "unfurled kite map", are shown to have such subalgebras in their cohomology of Milnor fibers or complements and subgroups in their link cohomology. In the case of general or skew-symmetric matrices, the results for the Milnor fibers and complements is valid for cohomology over $\mathbb{Z}$ (and hence any coefficient ring R); while for symmetric matrices, the results apply both for cohomology with coefficients in a field of characteristic zero or for $\mathbb{Z}/2\mathbb{Z}$-coefficients. In all three cases for a field of characteristic zero, cohomology subgroups are detected for the links which are above the middle dimensions.

Furthermore, we extend in §7 the results for complements and links for $m \times m$ matrices to general $m \times p$ matrices. This includes determining the form of the

characteristic cohomology and giving a detection criterion using an appropriate form of kite spaces and mappings.

A restricted form of the kite spaces serve a further purpose in §8 for identifying how the cohomology of local links of strata in the varieties of singular matrices relate to the cohomology of the global links.

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1. Matrix Equivalence for the Three Types of Matrix Singularities

We will apply the results in Part I [D6] to the cohomology for a matrix singularity $\mathcal{V}_0$ for any of the three types of matrices. We let M denote the space of $m \times m$ general matrices $M_m(\mathbb{C})$, resp. symmetric matrices $Sym_m(\mathbb{C})$, resp. skew-symmetric matrices $Sk_m(\mathbb{C})$ (for m even). We also let $\mathcal{D}_m^{(*)}$ denote the variety of singular matrices for each case with $(*)$ denoting $()$ for general matrices, (sy) for symmetric matrices, or (sk) for skew-symmetric matrices. For the case of $m \times p$ matrices with $m \neq p$ we use the notation $\mathcal{D}m, p \subset M_{m,p}(\mathbb{C})$. Also, the corresponding defining equations for the three $m \times m$ cases are given by: det for the general and symmetric cases and the Pfaffian Pf for the skew-symmetric case. We generally denote the defining equation by $H : \mathbb{C}^N, 0 \to \mathbb{C}, 0$ for $\mathcal{V}$, where $M \simeq \mathbb{C}^N$ for appropriate N in each case and $\mathcal{V} = \mathcal{D}_m^{(*)}$. For the case of $m \times p$ matrices with $m \neq p$, $\mathcal{D}m, p$ is not a hypersurface and we will not be concerned with its defining equation.

Abbreviated Notation for the Characteristic Cohomology.

For matrix singularities $\mathcal{V}_0$ defined by $f_0 : \mathbb{C}^n \to M, 0 = \mathbb{C}^N, 0$, the characteristic cohomology with coefficients R defined in Part I is denoted as follows: for the Milnor fiber (in the case $\mathcal{V}$ is a hypersurface) by $\mathcal{A}_{\mathcal{V}}(f_0, R)$, for the complement of $\mathcal{V}_0$ by $\mathcal{C}_{\mathcal{V}}(f_0, R)$, and for the link of $\mathcal{V}_0$, for $\mathbf{k}$ a field of characteristic 0, by $\mathcal{B}_{\mathcal{V}}(f_0, \mathbf{k})$. We use a simplified notation for matrix singularities.

$$\mathcal{A}_m^{(*)}(f_0; R) = \mathcal{A}_{\mathcal{D}_m^{(*)}}(f_0; R), \qquad \mathcal{C}^{(*)}(f_0; R) = \mathcal{C}_{\mathcal{D}_m^{(*)}}(f_0; R),$$

$$\text{and} \quad \mathcal{B}_m^{(*)}(f_0; \mathbf{k}) = \mathcal{B}_{\mathcal{D}_m^{(*)}}(f_0; \mathbf{k})$$

If m is understood, we shall suppress it in the notation.

For the case of $m \times p$ matrices with $m \neq p$, $\mathcal{D}_{m,p}$ is not a hypersurface, but we shall use for the complement and link

$$\mathcal{C}_{m,p}(f_0; R) \; = \; \mathcal{C}_{\mathcal{D}_{m,p}}(f_0; R) \quad \text{and} \quad \mathcal{B}_{m,p}(f_0; \mathbf{k}) \; = \; \mathcal{B}_{\mathcal{D}_{m,p}^{(*)}}(f_0; \mathbf{k})$$

Again, if (m, p) is understood, we shall suppress it in the notation.

Matrix Singularities Equivalences $\mathcal{K}_M$ and $\mathcal{K}_{HM}$.

There are several different equivalences that we shall consider for matrix singularities $f_0 : \mathbb{C}^n, 0 \to M, 0$ with $\mathcal{V}$ denoting the subvariety of singular matrices in M. The one used in classifications is $\mathcal{K}_M$–*equivalence:* We suppose that we are given an action of a group of matrices G on M. For symmetric or skew symmetric matrices, it is the action of $\mathrm{GL}_m(\mathbb{C})$ by $B \cdot A = B\, A\, B^T$. For general $m \times p$ matrices, it is the action of $\mathrm{GL}_m(\mathbb{C}) \times \mathrm{GL}_p(\mathbb{C})$ by $(B, C) \cdot A = B\, A\, C^{-1}$. Given such an action, then the group $\mathcal{K}_M$ consists of pairs (φ, B), with φ a germ of a diffeomorphism of $\mathbb{C}^n, 0$ and B a holomorphic germ $\mathbb{C}^n, 0 \to G, I$. The action is given by

$$f_0(x) \mapsto f_1(x) \; = \; B(x) \cdot (f_0 \circ \varphi^{-1}(x))\,.$$

Although $\mathcal{K}_M$ is a subgroup of $\mathcal{K}_{\mathcal{V}}$, they have the same tangent spaces and their path connected components of their orbits agree (for example this is explained in [DP, §2] because of the results due to Józefiak [J], Józefiak-Pragacz [JP], and Gulliksen-Negård[GN] as pointed out by Goryunov-Mond [GM]).

We next restrict to codimension 1 subgroups; let

$$GL_m(\mathbb{C})^{(2)} \; \overset{def}{=} \; \ker(\det \times \det : GL_m(\mathbb{C}) \times GL_m(\mathbb{C}) \to (\mathbb{C}^* \times \mathbb{C}^*)/\Delta\mathbb{C}^*)$$

where $\Delta\mathbb{C}^*$ is the diagonal subgroup. We then replace the groups for $\mathcal{K}_M$–equivalence by the subgroup $SL_m(\mathbb{C})$ for the symmetric and skew-symmetric case and for the general case the subgroup $GL_m(\mathbb{C})^{(2)}$. These restricted versions of equivalence preserve the defining equation H in each case. We denote the resulting equivalence groups by $\mathcal{K}_{HM}$, which are subgroups of the corresponding $\mathcal{K}_H$. As $\mathcal{K}_{HM}$ equivalences preserve H, they also preserve the Milnor fibers and the varieties of singular matrices $\mathcal{V}$. By the above referred to results, in each of the three cases, these $\mathcal{K}_{HM}$ also have the same tangent spaces as $\mathcal{K}_H$ in each case.

As a consequence of [D6, Prop. 2.1 and Prop. 2.2], since $\mathcal{K}_{HM}$ is a subgroup of $\mathcal{K}_H$, we have for any coefficient ring R and field $\mathbf{k}$ of characteristic 0 the following corollary.

Corollary 1.1. *For each of the three cases of the varieties of $m \times m$ singular matrices $\mathcal{V} = \mathcal{D}_m^{(*)}$, let $\mathcal{V}_0$ be defined by $f_0 : \mathbb{C}^n, 0 \to M, 0$, with M denoting the corresponding space of matrices. Then,*

  a) *the characteristic subalgebra $\mathcal{A}^{(*)}(f_0; R)$ is, up to Milnor fiber cohomology isomorphism, an invariant of the $\mathcal{K}_{HM}$–equivalence class of f_0;*
  b) *$\mathcal{B}^{(*)}(f_0; \mathbf{k})$ is, up an isomorphism of the cohomology of the link, an invariant of the $\mathcal{K}_M$–equivalence class of f_0; and*
  c) *the characteristic subalgebra $\mathcal{C}^{(*)}(f_0; R)$ is, up to an isomorphism of the cohomology of the complement, an invariant of the $\mathcal{K}_M$–equivalence class of f_0.*

Hence, the structure of the cohomology of the Milnor fiber of $\mathcal{V}_0$ as a graded algebra (or graded module) over $\mathcal{A}^{(*)}(f_0; R)$ is, up to isomorphism, independent of the $\mathcal{K}_{HM}$–equivalence class of f_0

Before considering the cohomology of the Milnor fibers of the $\mathcal{D}_m(*)$, we first give an important property which implies that each of the $\mathcal{D}_m^{(*)}$ are H-holonomic in the sense of [D2], which gives a geometric condition that assists in proving that the matrix singularity is finitely $\mathcal{K}_{HM}$-determined (and hence finitely $\mathcal{K}_H$-determined). This will be a consequence of the fact that for all three cases the above groups act transitively on the strata of the canonical Whitney stratification of $\mathcal{D}_m^{(*)}$.

Lemma 1.2. *For each of the three cases of $m \times m$ general, symmetric and skew-symmetric matrices, the corresponding subgroups $GL_m(\mathbb{C})^{(2)}$, resp. $SL_m(\mathbb{C})$ act transitively on the strata of the canonical Whitney stratification of $\mathcal{D}_m^{(*)}$.*

Proof of Lemma 1.2. First, for the general case, let $A \in \mathcal{D}_m$ have rank $r < m$. We also denote the linear transformation on the space of column vectors defined by A to be denoted by L_A. Then, we let $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ denote a basis for $\mathbb{C}^m$ so that $\{\mathbf{v}_{r+1}, \ldots, \mathbf{v}_m\}$ is a basis for $\ker(L_A)$. We also let $\{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$ denote a basis for $\mathbb{C}^m$ so that $\mathbf{w}_j = L_A(\mathbf{v}_j)$ for $j = 1, \ldots, r$. We let $b = \det(\mathbf{v}_1 \ldots \mathbf{v}_m)$ and $c = \det(\mathbf{w}_1 \ldots \mathbf{w}_m)$. Then, we let $B^{-1} = (\mathbf{v}_1, \ldots, \mathbf{v}_{m-1}, \frac{c}{b}\mathbf{v}_m)$ and $C^{-1} = (\mathbf{w}_1 \ldots \mathbf{w}_m)$. Then, $C \cdot A \cdot B^{-1} = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$, where I_r is the $r \times r$ identity matrix. Also,$\det(B) = \det(C) = c$ so $(B, C) \in GL_m(\mathbb{C})^{(2)}$. Thus, the each orbit of $GL_m(\mathbb{C}) \times GL_m(\mathbb{C})$, which consists of matrices of given fixed rank $< m$ is a stratum of the canonical Whitney stratification, is also an orbit of $GL_m(\mathbb{C})^{(2)}$.

For both the symmetric and skew-symmetric cases the corresponding orbits under $GL_m(\mathbb{C})$ consist of matrices of given symmetric or skew-symmetric type of fixed rank $< m$; and they form strata of the canonical Whitney stratification. We show that they are also orbits under the action of $SL_m(\mathbb{C})$.

For $A \in \mathcal{D}_m^{(sy)}$ of rank $r < m$, we consider the symmetric bilinear form $\psi(X, Y) = X^T \cdot A \cdot Y$ for column vectors in $\mathbb{C}^m$. We can find a basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ for $\mathbb{C}^m$ so that $\psi(\mathbf{v}_i, \mathbf{v}_i) = 1$ for $i = 1, \ldots, r$, $= 0$ for $i > r$, and $\psi(\mathbf{v}_i, \mathbf{v}_j) = 0$ if $i \neq j$. Then, let $b = \det(\mathbf{v}_1 \ldots \mathbf{v}_m)$, we let $B^T = (\mathbf{v}_1, \ldots, \mathbf{v}_{m-1}, \frac{1}{b}\mathbf{v}_m)$. Then $\det(B) = 1$ and $B \cdot A \cdot B^T = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$.

Lastly, for the skew-symmetric case the argument is similar, except for $A \in \mathcal{D}_m^{(sk)}$ of rank $r < m$, we consider the skew-symmetric bilinear form $\psi(X, Y) = X^T \cdot A \cdot Y$ for column vectors in $\mathbb{C}^m$ with even m and $r = 2k$. There is a basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ for $\mathbb{C}^m$ so that $\psi(\mathbf{v}_{2i-1}, \mathbf{v}_{2i}) = 1$ for $i = 1, \ldots, k$, and otherwise, $\psi(\mathbf{v}_i, \mathbf{v}_j) = 0$ for $i < j$. Then, let $b = \det(\mathbf{v}_1 \ldots \mathbf{v}_m)$, we let $B^T = (\mathbf{v}_1, \ldots, \mathbf{v}_{m-1}, \frac{1}{b}\mathbf{v}_m)$. Then $\det(B) = 1$ and $B \cdot A \cdot B^T = \begin{pmatrix} J_k & 0 \\ 0 & 0 \end{pmatrix}$, where J_k is the $r \times r$ block diagonal matrix with k 2×2-blocks of $J_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. $\qquad\square$

2. COHOMOLOGY OF THE MILNOR FIBERS OF THE $\mathcal{D}_m^{(*)}$

We next recall results from [D3] and [D4] giving the cohomology structure of the Milnor fibers of the $\mathcal{D}_m^{(*)}$ for each of the three types of matrices. This includes: representing the Milnor fibers by global Milnor fibers, giving compact symmetric spaces as compact models for the homotopy types of the global Milnor fibers, giving the resulting cohomology for the symmetric spaces, geometrically representing

the cohomology classes, and indicating the relation of the cohomology classes for different m.

Homotopy Type of Global Milnor fibers via Symmetric Spaces.

The global Milnor fibers for each of the three cases, which we denote by , F_m, resp. $F_m^{(sy)}$, resp. $F_m^{(sk)}$, are given by $H^{-1}(1)$ for $H : M, 0 \to \mathbb{C}, 0$ the defining equation for $\mathcal{D}_m^{(*)}$, which is det for the general of symmetric case and Pfaffian Pf for the skew-symmetric case. As shown in [D3] the Milnor fiber for the germ of H at 0 is diffeomorphic to the global Milnor fiber. We reproduce here the representation of the global Milnor fibers as a homogeneous spaces, whose homotopy types are given by symmetric spaces. These provide compact models for the Milnor fibers diffeomorphic to their Cartan models as given by [D4, Table 1]. these are given in Table 1.

Milnor Fiber $F_m^{(*)}$	Quotient Space	Symmetric Space	Compact Model $F_m^{(*)\,c}$	Cartan Model
F_m	$SL_m(\mathbb{C})$	SU_m	SU_m	F_m^c
$F_m^{(sy)}$	$SL_m(\mathbb{C})/SO_m(\mathbb{C})$	SU_m/SO_m	$SU_m \cap Sym_m(\mathbb{C})$	$F_m^{(sy)\,c}$
$F_m^{(sk)}, m = 2n$	$SL_{2n}(\mathbb{C})/Sp_n(\mathbb{C})$	SU_{2n}/Sp_n	$SU_m \cap Sk_m(\mathbb{C})$	$F_m^{(sk)\,c} \cdot J_n^{-1}$

TABLE 1. Global Milnor fiber, its representation as a homogeneous space, compact model as a symmetric space, compact model as subspace and Cartan model.

Tower Structures of Global Milnor fibers and Symmetric Spaces by Inclusion.

The global Milnor fibers for all cases, their symmetric spaces, and their compact models form towers via inclusions. These are given as follows. For the general and symmetric cases, there is the homomorphism $\tilde{j}_m : SL_m(\mathbb{C}) \hookrightarrow SL_{m+1}(\mathbb{C})$ sending $A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$. This can be identified with the inclusion of Milnor fibers $\tilde{j}_m : F_m \subset F_{m+1}$. Also, it restricts to give an inclusion $\tilde{j}_m : SU_m \hookrightarrow SU_{m+1}$ which are the compact models for the general case. Second, it induces an inclusion $\tilde{j}_m^{(sy)} : SL_m(\mathbb{C})/SO_m(\mathbb{C}) \hookrightarrow SL_{m+1}(\mathbb{C})/SO_{m+1}(\mathbb{C})$ which is an inclusion of Milnor fibers $\tilde{j}_m^{(sy)} : F_m^{(sy)} \hookrightarrow F_{m+1}^{(sy)}$. It also induces an inclusion of the compact homotopy models $\tilde{j}_m^{(sy)} : SU_m/SO_m(\mathbb{R}) \subset SU_{m+1}/SO_{m+1}(\mathbb{R})$ for the Milnor fibers.

For the skew symmetric case, the situation is slightly more subtle. First, the composition of two of the above successive inclusion homomorphisms for $SL_m(\mathbb{C})$ gives a homomorphism $SL_m(\mathbb{C}) \hookrightarrow SL_{m+2}(\mathbb{C})$ sending $A \mapsto \begin{pmatrix} A & 0 \\ 0 & I_2 \end{pmatrix}$ for the 2×2 identity matrix I_2. For even $m = 2k$, it induces an inclusion $\tilde{j}_m^{(sk)} : SL_m(\mathbb{C})/Sp_k(\mathbb{C}) \hookrightarrow SL_{m+2}(\mathbb{C})/Sp_{k+1}(\mathbb{C})$. However, the inclusion of Milnor fibers $\tilde{j}_m^{(sk)} : F_m^{(sk)} \hookrightarrow F_{m+2}^{(sk)}$ is given by the map sending $A \mapsto \begin{pmatrix} A & 0 \\ 0 & J_1 \end{pmatrix}$ for the 2×2 skew-symmetric matrix $J_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. These two inclusions are related via the action of $SL_m(\mathbb{C})/Sp_k(\mathbb{C})$

on $F_m^{(sk)}$ which induces a diffeomorphism given by $B \mapsto B \cdot J_k \cdot B^T$ $(m = 2k)$. This also induces an inclusion of compact homotopy models $SU_m \cap Sk_m(\mathbb{C}) \subset SU_{m+2} \cap Sk_{m+2}(\mathbb{C})$. This inclusion commutes with both the inclusion of the Milnor fibers under the diffeomorphism given in [D3] by the action, and the inclusion of the Cartan models induced from the compact models after multiplying by J_k^{-1}, see Table 1. The Schubert decompositions for all three cases given in [D4] satisfy the additional property that they respect the inclusions.

Cohomology of Global Milnor fibers using Symmetric Spaces.
    Next, we recall the form of the cohomology algebras for the global Milnor fibers. First, for the $m \times m$ matrices for the general case or skew-symmetric case (with $m = 2n$), by Theorems [D4, Thm. 6.1] and [D4, Thm. 6.14], the Milnor fiber cohomology with coefficients $R = \mathbb{Z}$ are given as follows.

$$(2.1) \qquad H^*(F_m; \mathbb{Z}) \simeq \Lambda^* \mathbb{Z}\langle e_3, e_5, \ldots, e_{2m-1}\rangle \quad \text{general case}$$

$$(2.2) \quad H^*(F_m^{(sk)}; \mathbb{Z}) \simeq \Lambda^* \mathbb{Z}\langle e_5, e_9, \ldots, e_{4n-3}\rangle \quad \text{skew-symmetric case } (m = 2n).$$

Therefore these isomorphisms continue to hold with $\mathbb{Z}$ replaced by any coefficient ring R. Thus, for any coefficient ring R, $\mathcal{A}^{(*)}(f_0; R)$ is the quotient ring of a free exterior R-algebra on generators e_{2j-1}, for $j = 2, 3, \ldots, m$, resp. e_{4j-3} for $j = 2, 3, \ldots, n$.
    For the $m \times m$ symmetric case there are two important cases where either $R = \mathbb{Z}/2\mathbb{Z}$ or $R = \mathbf{k}$, a field of characteristic zero. First, for the coefficient ring $R = \mathbf{k}$, the symmetric case breaks-up into two cases depending on whether m is even or odd (see [MT, Thm. 6.7 (2), Chap. 3] or Table 1 of [D3]).

$$(2.3) \qquad H^*(F_m^{(sy)}; \mathbf{k}) \simeq \begin{cases} \Lambda^* \mathbf{k}\langle e_5, e_9, \ldots, e_{2m-1}\rangle & \text{if } m = 2k+1 \\ \Lambda^* \mathbf{k}\langle e_5, e_9, \ldots, e_{2m-3}\rangle\{1, e_m\} & \text{if } m = 2k \end{cases}.$$

Here e_m is the Euler class of a m-dimensional real oriented vector bundle $\tilde{E}_m$ on the Milnor fiber $F_m^{(sy)}$. The vector bundle $\tilde{E}_m$ on the symmetric space $SU_m/SO_m(\mathbb{R})$ has the form $SU_m \times_{SO_m(\mathbb{R})} \mathbb{R}^m \to SU_m/SO_m(\mathbb{R})$ where the action of $SO_m(\mathbb{R})$ is given by the standard representation. This can be described as the *bundle of totally real subspaces* of $\mathbb{C}^m$, which is the bundle of m-dimensional real subspaces of $\mathbb{C}^m$ whose complexifications are $\mathbb{C}^m$.
    In the second case for $R = \mathbb{Z}/2\mathbb{Z}$, by Theorem [D4, Thm. 6.15] using [MT, Thm. 6.7 (3), Chap. 3], we have

$$(2.4) \qquad\qquad H^*(F_m^{(sy)}; \mathbb{Z}/2\mathbb{Z}) \simeq \Lambda^* \mathbb{Z}/2\mathbb{Z}\langle e_2, e_3, \ldots, e_m\rangle$$

for generators $e_j = w_j(\tilde{E}_m)$, for $j = 2, 3, \ldots, m$, for $w_j(\tilde{E}_m)$ the j-th Stiefel-Whitney class of the real oriented m-dimensional vector bundle $\tilde{E}_m$ above.
    We summarize the structure of the characteristic subalgebra $\mathcal{A}^{(*)}(f_0; R)$ in each case with the following.

Theorem 2.1. *Let $f_0 : \mathbb{C}^n, 0 \to M, 0$ define a matrix singularity $\mathcal{V}_0, 0$ for M the space of $m \times m$ matrices which are either general, symmetric, or skew-symmetric (with $m = 2n$).*

   i) *In the general and skew-symmetric cases, $\mathcal{A}^{(*)}(f_0; R)$ is a quotient of the free R-exterior algebra with generators given in* (2.1)

ii) *In the symmetric case with $R = \mathbb{Z}/2\mathbb{Z}$, $\mathcal{A}^{(sy)}(f_0; \mathbb{Z}/2\mathbb{Z})$ is the quotient of the free exterior algebra over $\mathbb{Z}/2\mathbb{Z}$ on generators $e_j = w_j(\tilde{E}_m)$, for $j = 2, 3, \ldots, m$, for $w_j(\tilde{E}_m)$ the Stiefel-Whitney classes of the real oriented m-dimensional vector bundle $\tilde{E}_m$ on the Milnor fiber of $\mathcal{D}_m^{(sy)}$. Hence, $\mathcal{A}^{(*)}(f_0; \mathbb{Z}/2\mathbb{Z})$ is a subalgebra generated by the Stiefel-Whitney classes of the pull-back vector bundle $f_{0,w}^*(\tilde{E}_m)$ on $\mathcal{V}_w$.*

iii) *In the symmetric case with $R = \mathbf{k}$, a field of characteristic zero, $\mathcal{A}^{(sy)}(f_0; \mathbf{k})$ is a quotient of the $\mathbf{k}$-algebras in each of the cases in (2.3).*

Then, in each of these cases, the cohomology (with coefficients in a ring R) of the Milnor fiber of $\mathcal{V}_0$ has a graded module structure over the characteristic subalgebra $\mathcal{A}^{(*)}(f_0; R)$ of f_0.

Cohomology Relations Under Inclusions for Varying m.

We give the relations between the cohomology of the global Milnor fibers and the symmetric spaces for varying m under the induced inclusion mappings. The relations are the following.

Proposition 2.2.   1) *In the general case, for the inclusions $\tilde{j}_{m-1} : SU_{m-1} \hookrightarrow SU_m$ and $\tilde{j}_{m-1} : F_{m-1} \subset F_m$, $\tilde{j}_{m-1}^*$ is an isomorphism on the subalgebra generated by $\{e_{2i-1} : i = 2, \ldots, m-1\}$ and $\tilde{j}_{m-1}^*(e_{2m-1}) = 0$.*

2) *In the skew-symmetric case (with $m = 2n$), for the inclusions $j_{m-2}^{(sk)} : SU_{2(n-1)}/Sp_{n-1} \hookrightarrow SU_{2n}/Sp_n$ and for Milnor fibers $\tilde{j}_{m-2}^{(sk)} : F_{m-2}^{(sk)} \hookrightarrow F_m^{(sk)}$, $\tilde{j}_{m-2}^{(sk)*}$ is an isomorphism on the subalgebra generated by $\{e_{4i-3} : i = 2, \ldots, m-1\}$ and $\tilde{j}_{m-2}^{(sk)*}(e_{4m-3}) = 0$.*

3) *In the symmetric case, for the inclusion $\tilde{j}_{m-1}^{(sy)} : SU_{m-1}/SO_{m-1}(\mathbb{R}) \hookrightarrow SU_m/SO_m(\mathbb{R})$ and for Milnor fibers $\tilde{j}_{m-1}^{(sy)} : F_{m-1}^{(sy)} \subset F_m^{(sy)}$:*

   i) *for coefficients $R = \mathbb{Z}/2\mathbb{Z}$, $\tilde{j}_{m-1}^{(sy)*}$ is an isomorphism on the subalgebra generated by $\{e_i : i = 2, \ldots, m-1\}$ and $\tilde{j}_{m-1}^{(sy)*}(e_m) = 0$;*

   iia) *for coefficients $R = \mathbf{k}$, a field of characteristic 0, if $m = 2k$, then $\tilde{j}_{m-1}^{(sy)*}$ is an isomorphism on the subalgebra generated by $\{e_{4i-3} : i = 2, \ldots, k\}$, and $\tilde{j}_{m-1}^{(sy)*}(e_m) = 0$, and*

   iib) *if $m = 2k+1$, then $\tilde{j}_{m-1}^{(sy)*}$ is an isomorphism on the subalgebra generated by $\{e_{4i-3} : i = 2, \ldots, k\}$, and $\tilde{j}_{m-1}^{(sy)*}(e_{2m-1}) = 0$.*

Proof. For the general and skew-symmetric cases, the Schubert decomposition for the Cartan models $\mathcal{C}_m$ and $\mathcal{C}_m^{(sk)}$ for successive m given in [D4] preserves the inclusions and the homology properties. In these two cases the result follows from the resulting identified Kronecker dual cohomology classes [D4, §6].

For the symmetric case and for $\mathbb{Z}/2\mathbb{Z}$-coefficients, an analogous Schubert decomposition gives the corresponding result. The remaining symmetric case for coefficients $\mathbf{k}$ a field of characteristic 0 does not follow in [D4] from the Schubert decomposition. Instead, the computation of the cohomology of the symmetric space given in [MT, Chap. 3] yields the result. In fact the algebraic computations in [MT, Chap. 3] (see also [Bo]) also give the results for the other cases.   □

Vanishing Compact Models for the Milnor Fibers of $\mathcal{D}_m^{(*)}$.

In part I we gave a detection criterion [D6, Lemma 3.2] for detecting the non-vanishing of a subgroup E of the characteristic cohomology of the Milnor fiber $\mathcal{A}^{(*)}(f_0, R)$. We did so using vanishing compact models for the Milnor fiber. We use the preceding compact models for the Milnor fibers to give vanishing compact models for detecting nonvanishing subalgebras of $\mathcal{A}^{(*)}(f_0, R)$. From the above, let $F_M^{(*),c}$ denote the compact models for the individual global Milnor fibers $F_M^{(*)}$. We define $\Phi : F_m^{(*),c} \times (0, 1] \to H^{-1}((0, 1])$ sending $\Phi(A, t) = t \cdot A$. Also, let $E = \Lambda^* R\{e_{i_1}, \ldots, e_{i_\ell}\}$ denote the exterior subalgebra of $H^*(F_m^{(*),c}; R)$ on generators of the ℓ lowest degrees. We also let $\lambda_E : F_\ell^{(*),c} \to F_m^{(*),c}$ denote the compositions $\tilde{\jmath}_{m-1}^{(*)} \circ \cdots \circ \tilde{\jmath}_\ell^{(*)}$. Then, by Proposition 2.2, λ_E^* induces an isomorphism from E to its image. Our goal is to first show that an appropriate restriction of Φ to a subinterval $(0, \delta)$ will provide a vanishing compact model for $F_M^{(*)}$; and moreover, we will use λ_E^* to give a germ which detects E. First, we give vanishing compact models for each case as follows.

Proposition 2.3. *A vanishing compact model for the Milnor fiber for $\mathcal{D}_M^{(*)}$ is given for sufficiently small $0 < \delta << \varepsilon$ by $\Phi : F_m^{(*),c} \times (0, \delta) \to H^{-1}((0, \varepsilon])$ sending $\Phi(A, t) = t \cdot A$.*

Proof. We begin by first making a few observations about the global Milnor fibers. For M one of the spaces of $m \times m$ matrices, we consider $H : M, 0 \to \mathbb{C}, 0$ the defining equation for $\mathcal{D}_m^{(*)}$ ($H = \det$ or Pf). Then, the global Milnor fiber is $F_m^{(*)} = H^{-1}(1)$. Now we can consider multiplication in M by a constant $a \neq 0$. As H is homogeneous, if $A \in F_m^{(*)}$, then $a \cdot A \in H^{-1}(a^m)$ in the general or symmetric cases, or in the skew-symmetric cases $H^{-1}(a^k)$ where $m = 2k$.

We also observe that multiplication by a is a diffeomorphism between these two Milnor fibers. We denote the image of $F_m^{(*)}$ by multiplication by a by $aF_m^{(*)}$. Then, by e.g. the proof of [D3, Lemma 1.2], given $\delta > 0$, there is an $a > 0$ so that $aF_m^{(*)} \cap B_\delta$ is the local Milnor fiber of $\mathcal{V}_0$, $aF_m^{(*)}$ is transverse to the spheres of radii $\geq \delta$, and $aF_m^{(*)} \cap B_\delta \subset aF_m^{(*)}$ is a homotopy equivalence.

Also, we have the compact homotopy models which occur as submanifolds of SU_m of the form SU_m for the general case, resp. $SU_m \cap Sym_m(\mathbb{C})$ for the symmetric case, resp. $SU_m \cap Sk_m(\mathbb{C})$ for the skew-symmetric case. Now, for the standard Euclidean norm on $M_n(\mathbb{C})$, $\|A\| = \sqrt{m}$ for $A \in SU_m$. Then, as well this holds for $SU_m \cap Sym_m(\mathbb{C})$, and for $SU_m \cap Sk_m(\mathbb{C})$. We denote the compact model in $F_m^{(*)}$ by $F_m^{(*)\,c}$. Then, in each case if $M \simeq \mathbb{C}^N$, $F_m^{(*)\,c} \subset S_{\sqrt{m}}^{2N-1}$, the sphere of radius $\sqrt{m}$. Thus, $aF_m^{(*)\,c} \subset S_{a\sqrt{m}}^{2N-1}$.

Then, we first choose $0 < \eta << \delta < 1$ so that $H : H^{-1}(B_\eta^*) \cap B_\delta \to B_\eta^*$ is the Milnor fibration of H.

We choose $0 < a < \eta$ so that also $a\sqrt{m} < \delta$. Then, we observe the composition $aF_m^{(*)\,c} \subset aF_m^{(*)} \cap B_\delta \subset aF_m^{(*)}$ is a homotopy equivalence. Hence, The restriction $\Phi : F_m^{(*)\,c} \times (0, a) \to H^{-1}(B_a^*) \cap B_\delta \to B_a^*$ restricts to a homotopy equivalence for each $0 < t < a$ and so gives a vanishing compact model. $\square$

In light of Theorem 2.1, there are several natural problems to be solved involving the characteristic cohomology for matrix singularities of each of the types.

Problems for the Characteristic Cohomology of the Milnor Fibers of Matrix Singularities:

1) Determine the characteristic subalgebras as the images of the exterior algebras by detecting which monomials map to nonzero elements in $H^*(\mathcal{V}_w; R)$.
2) Identify geometrically these non-zero monomials in 1) via the pull-backs of the Schubert classes.
3) For the symmetric case with $\mathbb{Z}/2\mathbb{Z}$-coefficients, compute the Stiefel-Whitney classes of the pull-back of the vector bundle $\tilde{E}_m$.
4) Determine a set of module generators for the cohomology of the Milnor fibers as modules over the characteristic subalgebras.

We will give partial answers to these problems in the next sections.

3. Kite Spaces of Matrices for Given Flag Structures

We begin by introducing for a flag of subspaces for $\mathbb{C}^m$, a *linear kite subspace* of size k in the space of $m \times m$ matrices of any of the three types: general $M_m(\mathbb{C})$, symmetric $Sym_m(\mathbb{C})$, or skew-symmetric $Sk_m(\mathbb{C})$ (with m even). We initially consider the standard flag for $\mathbb{C}^m$, given by $0 \subset \mathbb{C} \subset \mathbb{C}^2 \subset \cdots \subset \mathbb{C}^{m-1} \subset \mathbb{C}^m$. We choose coordinates $\{x_1, \cdots, x_m\}$ for $\mathbb{C}^m$ so that $\{x_1, \cdots, x_k\}$ are coordinates for $\mathbb{C}^k$ for each k.

We let $E_{i,j}$ denote the $m \times m$ matrix with entry 1 in the (i,j)-position and 0 otherwise. We also let $E_{i,j}^{(sy)} = E_{i,j} + E_{j,i}$, $i < j$, or $E_{i,i}^{(sy)} = E_{i,i}$ for the space of symmetric matrices; and $E_{i,j}^{(sk)} = E_{i,j} - E_{j,i}$, for $i < j$. Then, we define

Definition 3.1. For each of the three types of $m \times m$ matrices and the standard flag of subspaces of $\mathbb{C}^m$, the corresponding *linear kite subspace of size ℓ* is the linear subspace of the space of matrices defined as follows:

i) For $M_m(\mathbb{C})$, it is the linear subspace $\mathbf{K}_m(\ell)$ spanned by

$$\{E_{i,j} : 1 \le i, j \le \ell\} \cup \{E_{i,i} : \ell < i \le m\}$$

ii) For $Sym_m(\mathbb{C})$, it is the linear subspace $\mathbf{K}_m^{(sy)}(\ell)$ spanned by

$$\{E_{i,j}^{(sy)} : 1 \le i \le j \le \ell\} \cup \{E_{i,i}^{(sy)} : \ell < i \le m\}$$

iii) For $Sk_m(\mathbb{C})$ with m even, for ℓ also even, it is the linear subspace $\mathbf{K}_m^{(sk)}(\ell)$ spanned by

$$\{E_{i,j}^{(sk)} : 1 \le i < j \le \ell\} \cup \{E_{2i,2i+1}^{(sk)} : \ell < 2i < m\}$$

Furthermore, we refer to the germ of the inclusion $i_m^{(*)}(\ell) : \mathbf{K}_m^{(*)}(\ell), 0 \to M, 0$, for each of the three cases as a *linear kite map of size ℓ*.

The general form of elements "the kites" in the linear kite subspaces have the form given in (3.1).

$$(3.1) \qquad Q_{\ell, m-\ell} = \begin{pmatrix} A_\ell & 0_{\ell, m-\ell} \\ 0_{m-\ell, \ell} & D_{m-\ell} \end{pmatrix}$$

$$\begin{pmatrix} * & \cdots & * & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & 0 & \cdots & 0 \\ * & \cdots & * & 0 & \cdots & 0 \\ 0 & \cdots & 0 & * & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \ddots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & * \end{pmatrix}$$

FIGURE 1. Illustrating the form of elements of a linear kite space of size ℓ in either the space of general matrices or symmetric matrices. For general matrices the upper left matrix of size $\ell \times \ell$ is a general matrix, while for symmetric matrices it is symmetric.

$$\begin{pmatrix} * & \cdots & * & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & 0 & \cdots & 0 \\ * & \cdots & * & 0 & \cdots & 0 \\ 0 & \cdots & 0 & J_1(*) & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \ddots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & J_1(*) \end{pmatrix}$$

FIGURE 2. Illustrating the form of elements of a linear "skew-symmetric kite" space of size ℓ (with ℓ even) in the space of skew-symmetric matrices. The upper left $\ell \times \ell$ matrix is a skew-symmetric matrix.

where A_ℓ is an $\ell \times \ell$-matrix which denotes an arbitrary matrix in either $M_\ell(\mathbb{C})$, resp. $Sym_\ell(\mathbb{C})$, resp. $Sk_\ell(\mathbb{C})$; and $0_{r,s}$ denotes the zero $r \times s$ matrix. Also, $D_{m-\ell}$ denotes an arbitrary $(m-\ell) \times (m-\ell)$ diagonal matrix in the general or symmetric case as in Figure 1. In the skew symmetric case, $D_{m-\ell}$ denotes the 2×2 block diagonal matrix with skew-symmetric blocks of the form given by (3) as in Figure 2, with

$$J_1(*) = \begin{pmatrix} 0 & * \\ -* & 0 \end{pmatrix}$$

and " $*$ " denoting an arbitrary entry.

We next extend this to general flags, and then to nonlinear subspaces as follows. For each of the three types of matrices $M = M_m(\mathbb{C})$, resp. $Sym_m(\mathbb{C})$, resp. $Sk_m(\mathbb{C})$ (with m even).

Definition 3.2. An *unfurled kite map* of given matrix type is any element of the orbit of $i_m^{(*)}(\ell)$, for $(*) = ()$, resp. (sy), resp.(sk), under the corresponding equivalence group $\mathcal{K}_{HM}$.

A germ $f_0 : \mathbb{C}^n, 0 \to M, 0$ *contains a kite map of size* ℓ for each of the three cases if there is a germ of an embedding $g : \mathbf{K}_m^{(*)}(\ell), 0 \to \mathbb{C}^n, 0$ such that $f_0 \circ g$ is an unfurled kite map.

Remark 3.3. We note that unfurled kite maps have the property that the standard flag can be replaced by a general flag; and moreover, the flag and linear kite

$$\begin{pmatrix} x_{1,1} + 2x_{4,4}x_{1,3} & x_{1,2} + 2x_{4,4}x_{2,3} & x_{1,3} + 2x_{4,4}x_{3,3} & 0 \\ x_{1,2} + 2x_{4,4}x_{2,3} & x_{2,2} - 35x_{1,2}x_{4,4} & x_{2,3} & (7x_{1,2} - 5)x_{4,4} \\ x_{1,3} + 2x_{4,4}x_{3,3} & x_{2,3} & x_{3,3} & 0 \\ 0 & (7x_{1,2} - 5)x_{4,4} & 0 & x_{4,4} \end{pmatrix}$$

FIGURE 3. An example of an unfurled kite map of size 3 into 4×4 symmetric matrices.

space can undergo nonlinear deformations. These can be performed by iteratively applying appropriate row and column operations using elements of the local ring of germs on $\mathbb{C}^n, 0$ instead of constants.

A simple example of an unfurled kite map is given in Figure 3.

4. DETECTING CHARACTERISTIC COHOMOLOGY USING KITE SPACES OF MATRICES

In §3 of Part I, we gave a detection criterion [D6, Lemma 3.2] for detecting the nonvanishing of a subgroup E of the characteristic cohomology of the Milnor fiber $\mathcal{A}^{(*)}(f_0, R)$. In this section we use this criterion using kite maps to detect nonvanishing exterior subalgebras of $\mathcal{A}^{(*)}(f_0, R)$. In §2, we gave in equations (2.1), (2.3), and (2.4) the cohomology of the Milnor fibers for the $\mathcal{D}_m^{(*)}$ for each of the three types of matrices. Thus, as for any matrix singularity $f_0 : \mathbb{C}^n, 0 \to M, 0$, by Theorem 2.1 the characteristic subalgebra is a quotient of the corresponding algebra. We let $E = \Lambda^* R\{e_{i_1}, \dots, e_{i_\ell}\} \subseteq H^*(F_m^{(*),c}; R)$ denote an exterior algebra on generators of the ℓ lowest degrees. Then, using the map λ_E given before Proposition 2.3, λ_E^* induces an isomorphism from E to its image. We next use λ_E to show that for germs f_0 containing a kite map of size ℓ for each case detects E in $\mathcal{A}^{(*)}(f_0, R)$.

Theorem 4.1. *Let $f_0 : \mathbb{C}^n, 0 \to M, 0$ define an $m \times m$ matrix singularity of one of the three types.*

a) *In the case of general matrices, if f_0 contains an unfurled kite map of size $\ell < m$, then $\mathcal{A}(f_0, R)$ contains an exterior algebra of the form*

$$\Lambda^* R\langle e_3, e_5, \dots, e_{2\ell-1}\rangle.$$

   on $\ell - 1$ generators.

b) *In the case of skew-symmetric matrices (with m even), if f_0 contains an unfurled skew-symmetric kite map of size $\ell(= 2k) < m$, then $\mathcal{A}^{(sk)}(f_0, R)$ contains an exterior algebra of the form*

$$\Lambda^* R\langle e_5, e_9, \dots, e_{4k-3}\rangle.$$

   on $k - 1$ generators.

c) *In the case of symmetric matrices, if f_0 contains an unfurled symmetric kite map of size $\ell < m$, then $\mathcal{A}^{(sy)}(f_0, R)$ contains an exterior algebra of one of the forms*

$$\Lambda^* \mathbf{k}\langle e_3, e_5, \dots, e_{2\ell-1}\rangle \qquad \text{if } R = \mathbf{k} \text{ is a field of characteristic } 0,$$
$$\Lambda^* \mathbb{Z}/2\mathbb{Z}\langle e_2, e_3, \dots, e_\ell\rangle \qquad \text{if } R = \mathbb{Z}/2\mathbb{Z},$$

Remark 4.2. In the symmetric case, it follows from c) that if f_0 contains an unfurled symmetric kite map of size $\ell < m$, then the Stiefel-Whitney classes of the pull-back bundle $w_i(f_{0,w}^*(\tilde{E}_m))$ on $\mathcal{V}_w$ are non-vanishing for $i = 2, \dots, \ell$.

Proof. By Theorem 2.1 and the Detection Lemma [D6, Lemma 3.2], it is sufficient to show that the corresponding kite maps of each type detect the corresponding exterior subalgebra. We use the notation from the proof of Proposition 2.3 which gave the vanishing compact models for the Milnor fibers in each case.

Then, we choose $0 < \eta << \varepsilon < \delta < 1$ so that $H : H^{-1}(B_\eta^*) \cap B_\delta \to B_\eta^*$ is the Milnor fibration of H and $H \circ i_m^{(*)}(\ell) : (H \circ i_m^{(*)}(\ell))^{-1}(B_\eta^*) \cap B_\varepsilon \to B_\eta^*$ is the Milnor fibration of $H \circ i_m^{(*)}(\ell)$. We also choose $0 < a < \eta$ so that $a\sqrt{m} < \varepsilon$.

Then there are the following inclusions.

$$(4.1) \qquad aF_\ell^{(*)\,c} \subset i_m^{(*)}(\ell)(H \circ i_m^{(*)}(\ell))^{-1}(a^r) \cap B_\varepsilon) \subset aF_m^{(*)} \cap B_\varepsilon \subset aF_m^{(*)},$$

where $r = m$ in the general or symmetric case or $r = \frac{m}{2}$ in the skew-symmetric case. The composition of inclusions $F_\ell^{(*)\,c} \subset F_m^{(*)\,c} \subset F_m^{(*)}$ commutes with multiplication by a as in (4.2) where each vertical map is a diffeomorphism given by multiplication by a.

$$(4.2) \qquad
\begin{array}{ccccc}
F_\ell^{(*)\,c} & \longrightarrow & F_m^{(*)\,c} & \longrightarrow & F_m^{(*)} \\
\downarrow & & \downarrow & & \downarrow \\
aF_\ell^{(*)\,c} & \xrightarrow{\;i_m^{(*)}(\ell)\;} & aF_m^{(*)\,c} & \longrightarrow & aF_m^{(*)}
\end{array}$$

Also, $i_m^{(*)}(\ell)$ in the bottom row is given by the map in (4.3).

$$(4.3) \qquad aA \mapsto aQ_{\ell,m-\ell} = \begin{pmatrix} aA_\ell & 0_{\ell,m-\ell} \\ 0_{m-\ell,\ell} & aD_{m-\ell} \end{pmatrix}$$

Then, by Proposition 2.2 the induced homomorphisms in cohomology for the top row of (4.2) restrict to an isomorphism on the corresponding exterior subalgebra of $H^*(F_m^{(*)}; R)$ onto the cohomology $H^*(F_\ell^{(*)\,c}; R)$, and vanishing on the remaining generators. Hence, as the vertical diffeomorphisms induce isomorphisms on cohomology, the induced homomorphisms on cohomology for the bottom row have the same property. Lastly, in (4.1), the induced homomorphisms in cohomology restrict to an isomorphism on the corresponding exterior subalgebra of $H^*(F_m^{(*)\,c}; R)$ to $H^*(aF_\ell^{(*)\,c}; R)$. Thus the induced homomorphism to the Milnor fiber of $H \circ i_m^{(*)}(\ell)$,

$$H^*(aF_m^{(*)\,c}; R) \longrightarrow H^*(H \circ i_m^{(*)}(\ell))^{-1}(a^r) \cap B_\varepsilon; R)$$

restricts to an isomorphism of the corresponding exterior algebra onto its image. Thus, the cohomology of the Milnor fiber of $H \circ i_m^{(*)}(\ell)$ contains the claimed exterior subalgebra. Thus, the flag map $i_m^{(*)}(\ell)$ detects the corresponding exterior algebra, so the result follows by the Detection Lemma. $\qquad\square$

5. Examples of Matrix Singularities Exhibiting Characteristic Cohomology

We consider several examples illustrating Theorem 4.1.

$$\begin{pmatrix} x_{1,1} + 2x_{4,4}x_{1,3} & x_{1,2} + 2x_{4,4}x_{2,3} & x_{1,3} + 2x_{4,4}x_{3,3} & y_1 \\ x_{1,2} + 2x_{4,4}x_{2,3} & x_{2,2} - 35x_{1,2}x_{4,4} & x_{2,3} + y_1x_{1,1}^2 & (7x_{1,2} - 5)x_{4,4} \\ x_{1,3} + 2x_{4,4}x_{3,3} & x_{2,3} + y_1x_{1,1}^2 & x_{3,3} + y_2x_{2,2}^2 & y_2 \\ y_1 & (7x_{1,2} - 5)x_{4,4} & y_2 & x_{4,4} \end{pmatrix}$$

FIGURE 4. An example of a germ f_0 containing an unfurled kite map of size 3 into 4×4 symmetric matrices in Figure 3.

Example 5.1. Let $f_0; \mathbb{C}^9, 0 \to Sym_4(\mathbb{C}), 0$ be defined by $f_0(\mathbf{x}, \mathbf{y})$ given by the matrix in Figure 4 for $\mathbf{x} = (x_{1,1}, x_{1,2}, x_{1,3}, x_{2,2}, x_{2,3}, x_{3,3}, x_{4,4})$ and $\mathbf{y} = (y_1, y_2)$. We let $\mathcal{V}_0 = f_0^{-1}(\mathcal{D}_4^{(sy)})$. This is given by the determinant of the matrix in Figure 4 defining f_0. Then, $\mathcal{V}_0$ has singularities in codimension 2. We observe that when $\mathbf{y} = (0, 0)$ we obtain the unfurled kite map in Figure 3. Thus, by Theorem 4.1, the Milnor fiber of $\mathcal{V}_0$ has cohomology with $\mathbb{Z}/2\mathbb{Z}$ coefficients containing the subalgebra $\Lambda^*\mathbb{Z}/2\mathbb{Z}\langle e_2, e_3\rangle$, so there is $\mathbb{Z}/2\mathbb{Z}$ cohomology in degrees 2, 3, and 5. We also note that $e_j = w_j(f_{0,w}^* \tilde{E}_4)$ so that one consequence is that the second and third Stiefel-Whitney classes of the pullback of the vector bundle $\tilde{E}_4$ are non-zero.

For coefficients a field $\mathbf{k}$ of characteristic 0, the cohomology of the Milnor fiber of $\mathcal{V}_0$ has an exterior algebra $\Lambda^*\mathbf{k}\langle e_5\rangle$, so there is a $\mathbf{k}$ generator e_5 in degree 5.

By Kato-Matsumota [KM], as singularities have codimension 2, the Milnor fiber is simply connected. Then, we can use the preceding to deduce information about the integral cohomology of the Milnor fiber from the universal coefficient theorem. It must have rank at least 1 in dimension 5, and it has 2-torsion in dimension 2.

Second, we consider a general matrix singularity.

$$\begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} & g_1(\mathbf{x}, \mathbf{y}) \\ x_{2,1} & x_{2,2} & x_{2,3} & x_{2,4} & g_2(\mathbf{x}, \mathbf{y}) \\ x_{3,1} & x_{3,2} & x_{3,3} & x_{3,4} & g_3(\mathbf{x}, \mathbf{y}) \\ x_{4,1} & x_{4,2} & x_{4,3} & x_{4,4} & g_4(\mathbf{x}, \mathbf{y}) \\ y_1 & y_2 & y_3 & y_4 & x_{5,5} \end{pmatrix}$$

FIGURE 5. An example of a germ f_0 in Example 5.2, containing a linear kite map of size 4 into 5×5 general matrices with $g_i(\mathbf{x}, 0) \equiv 0$ for each i).

Example 5.2. We let $f_0; \mathbb{C}^{21}, 0 \to M_5(\mathbb{C}), 0$ be defined with $f_0(\mathbf{x}, \mathbf{y})$ given by the matrix in Figure 5 for $\mathbf{x} = (x_{1,1}, \ldots, x_{4,4}, x_{5,5})$ and $\mathbf{y} = (y_1, y_2, y_3, y_4)$. In this example we require that $g_i(\mathbf{x}, 0) \equiv 0$ for each i . We let $\mathcal{V}_0 = f_0^{-1}(\mathcal{D}_5)$. This is given by the determinant of the matrix in Figure 5 defining f_0. Then, the $\mathcal{V}_0$ has singularities in codimension 4 in $\mathbb{C}^{21}$; hence by Kato-Matsumoto, the Milnor fiber is 2-connected. We observe that when $\mathbf{y} = (0, 0, 0, 0)$ we obtain the linear kite map $i_5(4)$. Thus, by Theorem 4.1, the Milnor fiber of $\mathcal{V}_0$ has characteristic cohomology with integer coefficients containing the subalgebra $\Lambda^*\mathbb{Z}\langle e_3, e_5, e_7\rangle$. Hence, the integer cohomology has rank at least 1 in dimensions $0, 3, 5, 7, 8, 10, 12, 15$. We cannot determine at this point whether the generator e_9 maps to a nonzero element in the cohomology of the Milnor fiber of $\mathcal{V}_0$. Even if it does, there are several products

involving e_9 in exterior algebra for the cohomology of $\mathcal{D}_5$ must map to 0, as the Milnor fiber is homotopy equivalent to a CW-complex of dimension 20.

Module structure over $\mathcal{A}^{(*)}(f_0, R)$ for the Cohomology of the Milnor fiber.

Remark 5.3. In §4 of part I, we considered how in the hypersurface case the cohomology of the Milnor fiber is a module over the characteristic cohomology and listed four issues which must be addressed. Already for condition i) and $\mathcal{V} = \mathcal{D}_m^{(*)}$, this leaves the remaining issues to be addressed:

  1) giving a sufficient condition that guarantees that the partial criterion [D6, (4.2)] is satisfied to ensure that for the singular Milnor number $\mu_{\mathcal{V}}(f_0)$ there is a contribution of a summand of that rank..

  2) determining $\mu_{\mathcal{V}}(f_0)$ for $\mathcal{V} = \mathcal{D}_m^{(*)}$. In the case that $\mathcal{V}_0$ has an isolated singularity (which requires that n is small, i.e. $n \leq \mathrm{codim}(\mathrm{sing}(\mathcal{D}_m^{(*)}))$, but allows arbitrary m), Goryunov-Mond [GM] give a formula in all three cases for $\mu_{\mathcal{V}}(f_0)$ in terms of the formula of [DM] for free divisors with a correction term given by an Euler characteristic for a Tor sequence. Alternatively, by a different method using "free completions" in all three cases, with arbitrary n but for small m, Damon-Pike [DP] give formulas for $\mu_{\mathcal{V}}(f_0)$ as alternating sums of lengths of explicit determinantal modules. However, there still does not exist a formula valid for all m and n.

6. Characteristic Cohomology for the Complements and Links of Matrix Singularities

We now turn to the characteristic cohomology of the complement and link for matrix singularities of all three types. Again, we may apply the Second Detection Lemma of Part I [D6, Lemma 3.4] for complements to detect a nonvanishing subalgebra of $\mathcal{C}^{(*)}(f_0, R)$ and corresponding nonvanishing subgroups of $\mathcal{B}^{(*)}(f_0, \mathbf{k})$. In order to apply the earlier results to the cases of matrix singularities, we first recall in Table 2 the cohomology, with coefficients a field $\mathbf{k}$ of characteristic 0, of the complements and links as given in [D3, table 2]. We will then use the presence of kite maps to detect both subalgbras of $\mathcal{C}^{(*)}(f_0, R)$ for the complements and subgroups of $\mathcal{B}^{(*)}(f_0, \mathbf{k})$ for the links.

Theorem 6.1. *Let $f_0 : \mathbb{C}^n, 0 \to M, 0$ define a matrix singularity $\mathcal{V}_0$ of any of the three types. If f_0 contains a kite map of size ℓ, then the characteristic cohomology of the complement $\mathcal{C}^{(*)}(f_0, \mathbf{k})$, for a field $\mathbf{k}$ of characteristic 0, contains an exterior algebra given by Table 3.*

Furthermore, the characteristic cohomology of the link $\mathcal{B}^{(*)}(f_0, \mathbf{k})$, as a graded vector space contains the graded subspace given by truncating the exterior subalgebra of $\mathcal{C}^{(*)}(f_0, \mathbf{k})$ listed in column 2 of Table 3 in the top degree and shifting by the amount listed in the last column.

For the complements in the general and skew-symmetric cases, $\mathbf{k}$ may be replaced by any coefficient ring R.

Remark 6.2. In what follows to simplify statements, instead of referring to the complement of $\mathcal{V}_0, 0 \subset \mathbb{C}^n, 0$ as $B_\varepsilon \backslash \mathcal{V}_0$ for sufficiently small $\varepsilon > 0$, we will just refer to the complement as $\mathbb{C}^n \backslash \mathcal{V}_0$, with the understanding that it is restricted to a sufficiently small ball.

Determinantal Hypersurface	Complement $M\backslash\mathcal{D}$	$H^*(M\backslash\mathcal{D}, \mathbf{k}) \simeq$ $H^*(K/L, \mathbf{k})$	Shift
$\mathcal{D}_m^{sy}$ (m = 2k+1)	$GL_m(\mathbb{C})/O_m(\mathbb{C})$ $\sim U_m/O_m(\mathbb{R})$	$\Lambda^*\mathbf{k}\langle e_1, e_5, \ldots, e_{2m-1}\rangle$	$\binom{m+1}{2} - 2$
$\mathcal{D}_m^{sy}$ (m = 2k)	$GL_m(\mathbb{C})/O_m(\mathbb{C})$	$\Lambda^*\mathbf{k}\langle e_1, e_5, \ldots, e_{2m-3}\rangle$	$\binom{m+1}{2} + m - 2$
$\mathcal{D}_m$	$GL_m(\mathbb{C}) \sim U_m$	$\Lambda^*\mathbf{k}\langle e_1, e_3, \ldots, e_{2m-1}\rangle$	$m^2 - 2$
$\mathcal{D}_m^{sk}$ (m = 2k)	$GL_{2k}(\mathbb{C})/Sp_k(\mathbb{C})$ $\sim U_{2k}/Sp_k$	$\Lambda^*\mathbf{k}\langle e_1, e_5, \ldots, e_{2m-3}\rangle$	$\binom{m}{2} - 2$

TABLE 2. The cohomology of the complements $M\backslash\mathcal{D}$ and links $L(\mathcal{D})$ for each determinantal hypersurface $\mathcal{D}$. The complements, are homotopy equivalent to the quotients of maximal compact subgroups K/L with cohomology given in the third column, where the generators of the cohomology e_k are in degree k; and the structure is an exterior algebra. For the links $L(\mathcal{D})$, the cohomology is isomorphic as a vector space to the cohomology of the complement truncated in the top degree and shifted by the degree indicated in the last column.

Proof of Theorem 6.1. The proof is similar to that for Theorem 4.1. As the statements are independent of f_0 in a given $\mathcal{K}_\mathcal{V}$-equivalence class, we may apply an element of $\mathcal{K}_H$ to obtain an f_0 containing a linear kite map. It is sufficient to show, as for the case of Milnor fibers, that the linear kite map detects the indicated subalgebra in $\mathcal{C}^{(*)}(f_0, \mathbf{k})$, and then apply Alexander duality for the result for the link.

By the results in [D3] summarized in Table 2, the complement $M\backslash\mathcal{D}_m^{(*)}$ is given by a homogeneous space G/H which has as a compact homotopy model (K/L) where $K = U_m$ for each of the cases. For successive values of m, we have for the three cases the successive inclusions:

i) for the general case, $GL_m(\mathbb{C}) \hookrightarrow GL_{m+1}(\mathbb{C})$ by $A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$;

ii) for the symmetric case, $GL_m(\mathbb{C}) \hookrightarrow GL_{m+1}(\mathbb{C})$ sending $A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$ induces an inclusion $GL_m(\mathbb{C})/O_m(\mathbb{C}) \hookrightarrow GL_{m+1}(\mathbb{C})/O_{m+1}(\mathbb{C})$;

iii) for even $m = 2k$, $GL_m(\mathbb{C}) \hookrightarrow GL_{m+2}(\mathbb{C})$ sending $A \mapsto \begin{pmatrix} A & 0 \\ 0 & I_2 \end{pmatrix}$, for I_2 the 2×2 identity matrix, induces an inclusion $GL_m(\mathbb{C})/Sp_k(\mathbb{C}) \hookrightarrow GL_{m+2}(\mathbb{C})/Sp_{k+1}(\mathbb{C})$.

Then, these are obtained by the action of $GL_m(\mathbb{C})$ on the appropriate spaces of matrices. They restrict to the compact homogenenous spaces which are homotopy equivalent models for the complements, given in Table 2 and which we denote by K/L for each of the three cases. Also, the inclusions correspond to the following inclusions of spaces of matrices.

i) for the general case, $M_m(\mathbb{C}) \hookrightarrow M_{m+1}(\mathbb{C})$ by $A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$;

ii) for the symmetric case, $Sym_m(\mathbb{C}) \hookrightarrow Sym_{m+1}(\mathbb{C})$ sending $A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$;

iii) for even $m = 2k$, $Sk_m(\mathbb{C}) \hookrightarrow Sk_{m+2}(\mathbb{C})$ sending $A \mapsto \begin{pmatrix} A & 0 \\ 0 & J_1 \end{pmatrix}$, for the

2×2 skew-symmetric matrix $J_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Furthermore, for the cohomology of these spaces (via their homotopy equivalent compact models K/L for each case) the maps induced by the inclusions sends $e_j \mapsto e_j$ for the nonzero generators in successive spaces.

Via these inclusions, the corresponding actions of $GL_m(\mathbb{C})$ on these spaces (as explained in [D3]) applied to either I_m for the general or symmetric case, or J_k for the skew symmetric case factor through the homogeneous spaces given in Table 2 to give diffeomorphisms to the complements of $\mathcal{D}_m^{(*)}$ in each case. The inclusions of the homogeneous spaces correspond to the inclusions of the spaces of nonsingular matrices. Under this correspondence, the cohomology of the homogeneous spaces gives the cohomology of the complements of the spaces of $m \times m$ singular matrices $M_m^{(*)} \backslash \mathcal{D}_m^{(*)}$. Here we let $M_m^{(*)}$ denotes the space of $m \times m$ matrices of appropriate type.

Just as for Milnor fibers, we use multiplication to define a vanishing compact model for the complement. We let $\mathcal{P}^{(*)} \subset M_m^{(*)} \backslash \mathcal{D}_m^{(*)}$ denote the compact model for the complement in each of the three cases. The action of U_m in each case gives elements A of the compact model to be products of elements of U_m and hence $\|A\| = \sqrt{m}$. Thus, $\mathcal{P}^{(*)} \subset B_{\sqrt{m}}$. Then, we can multiply the spaces of matrices by nonzero constants a and for each case $a \cdot \mathcal{P}^{(*)} \subset B_{a\sqrt{m}}$. Then, for a neighborhood B_δ of 0 in $M_m^{(*)}$, if $a\sqrt{(m)} < \delta$, then $a \cdot \mathcal{P}^{(*)} \subset B_\delta \backslash \mathcal{D}_m^{(*)}$.

Then, we define $\Phi : \mathcal{P}^{(*)} \times (0, a) \to M_m^{(*)} \backslash \mathcal{D}_m^{(*)}$ sending $\Phi(A, t) = t \cdot A$. Then, Φ defines a vanishing compact model for the complement for each case.

It remains to show that the kite map of size ℓ detects the corresponding exterior algebra given in Table 3 for the characteristic cohomology of the complement. We consider $i_m^{(*)}(\ell) : \mathbf{K}_\ell(\mathbb{C}) \cap B_\varepsilon \to M_m^{(*)}$. If $M_\ell^{(*)}$ denotes the embedding of the corresponding $\ell \times \ell$ matrices given above, then there is an $a > 0$ so that $aM_\ell^{(*)} \subset \mathbf{K}_m(\ell) \cap B_\varepsilon$. Then, as in the proof of Theorem 4.1, the composition

$$a(M_\ell^{(*)} \backslash \mathcal{D}_\ell^{(*)}) \subset (\mathbf{K}_m(\ell) \backslash \mathcal{D}_m^{(*)}) \cap B_\varepsilon \xrightarrow{i_m^{(*)}(\ell)} M_m^{(*)} \backslash \mathcal{D}_m^{(*)}$$

induces in cohomology an isomorphism from the exterior subalgebra given in Table 3 to a subalgebra of the cohomology of $a(M_\ell^{(*)} \backslash \mathcal{D}_\ell^{(*)})$ (since it is diffeomorphic to $M_\ell^{(*)} \backslash \mathcal{D}_\ell^{(*)}$). As this homomorphism factors through $H^*(\mathbb{C}^n \backslash \mathcal{V}_0; \mathbf{k})$, it is also an isomorphism onto a subalgebra of this cohomology. This shows that $i_m^{(*)}(\ell)$ detects the exterior algebra, so by the Second Detection lemma in Part I [D6, Lemma 3.4], the result follows for the complement.

Lastly, let $\widetilde{\Gamma}^{(*)}(f_0, \mathbf{k})$ denote the graded subspace of reduced homology obtained from the Kronecker dual $\Gamma^{(*)}(f_0, \mathbf{k})$ to this subalgebra. Then, by Alexander duality we obtain a graded subspace of $H^*(L(\mathcal{V}_0); \mathbf{k})$ isomorphic to $\widetilde{\Gamma}^{(*)}(f_0, \mathbf{k})$. It remains to show it is obtained from the exterior algebra by truncating it and applying an appropriate shift. As the exterior algebra satisfies Poincare duality under multiplication, this is done using the same argument in the proof of [D3, Prop. 1.9]. $\square$

We reconsider the examples from §5

Determinantal Hypersurface Type	$\mathcal{C}^{(*)}(f_0, \mathbf{k})$ contains subalgebra	Shift for Link
$\mathcal{D}_m^{sy}$ ℓ odd	$\Lambda^*\mathbf{k}\langle e_1, e_5, \ldots, e_{2\ell-1}\rangle$	$2n - \binom{\ell+1}{2} - 2$
$\mathcal{D}_m^{sy}$ ℓ even	$\Lambda^*\mathbf{k}\langle e_1, e_5, \ldots, e_{2\ell-3}\rangle$	$2n - \binom{\ell}{2} - 2$
$\mathcal{D}_m$	$\Lambda^*\mathbf{k}\langle e_1, e_3, \ldots, e_{2\ell-1}\rangle$	$2n - \ell^2 - 2$
$\mathcal{D}_m^{sk}$ (m = 2k) ℓ even	$\Lambda^*\mathbf{k}\langle e_1, e_5, \ldots, e_{2\ell-3}\rangle$	$2n - \binom{\ell}{2} - 2$

TABLE 3. The characteristic cohomology with coefficients in a field $\mathbf{k}$ of characteristic 0 for $\mathcal{V}_0 = f_0^{-1}(\mathcal{V})$ for each matrix type $\mathcal{V} = \mathcal{D}_m^{(*)}$. If f_0 contains an unfurled kite map of size ℓ, the characteristic cohomology $\mathcal{C}^{(*)}(f_0, \mathbf{k})$ contains an exterior subalgebra given in column 2 (where e_j has degree j). Then, for the link $L(\mathcal{V}_0)$, the characteristic cohomology contains as a graded subspace the exterior algebra in column 2 truncated in the top degree and shifted by the degree indicated in the last column. For the complements in the general or skew-symmetric cases, $\mathbf{k}$ in column 2 may be replaced by any coefficient ring R.

Example 6.3. In Example 5.1, we considered a singularity $\mathcal{V}_0$ defined by $f_0; \mathbb{C}^9, 0 \to Sym_4(\mathbb{C}), 0$ given by the matrix in Figure 4. It contains an unfurled kite map of size 3. We can apply Theorem 6.1.

For coefficients a field $\mathbf{k}$ of characteristic 0, from Table 3 the characteristic cohomology of the complement of $\mathcal{C}^{(sy)}(f_0, \mathbf{k})$ contains an exterior algebra $\Lambda^*\mathbf{k}\langle e_1, e_5\rangle$, so there are $\mathbf{k}$-vector space generators 1, e_1, e_5, and $e_1 \cdot e_5$ in degrees 0, 1, 5 and 6.

The characteristic cohomology $\mathcal{B}^{(sy)}(f_0, \mathbf{k})$ of the link of $\mathcal{V}_0$, contains the subspace obtained by upper truncating the exterior algebra to obtain the $\mathbf{k}$ vector space $\mathbf{k}\langle 1, e_1, e_5\rangle$ and shifting by $2 \cdot 9 - 2 - \binom{4}{2} = 10$ to obtain 1-dimensional generators in degrees 10, 11, and 15. We note that the Link $L(\mathcal{V}_0)$ has real dimension 15, so a vector space generator of the characteristic subalgebra generates the top dimensional class.

We also note that from Table 2 that $\mathcal{D}_4^{(sy)}$ has link cohomology given by the upper truncated $\Lambda^*\mathbf{k}\langle e_1, e_5\rangle$ but shifted by $\binom{5}{2} + 4 - 2 = 12$ so there is 1 dimensional cohomology in degrees 12, 13, and 17. Thus, f_0^* does not send any of these classes to nonzero classes in the characteristic cohomology.

We do note that for the kite map $i_4^{(sy)}(3) : \mathbb{C}^7, 0 \to Sym_4(\mathbb{C}), 0$ the characteristic cohomology for the link is the upper truncated exterior algebra giving the $\mathbf{k}$ vector space $\mathbf{k}\langle 1, e_1, e_5\rangle$ and then shifted by 6. Thus, its degrees are 6, 7 and 11. We see that as noted in [D6, Remark 1.8] in terms of the relative Gysin homomorphism, there is a shift in degrees given by twice the difference in dimension between each of the maps.

Second, we return to Example 5.2.

Example 6.4. From Example 5.2, the singularity $\mathcal{V}_0 = f_0^{-1}(\mathcal{D}_5)$ is defined by $f_0; \mathbb{C}^{21}, 0 \to M_5(\mathbb{C}), 0$, given by the matrix in Figure 5. Also, f_0 contains the linear

kite map of size 4. Thus, we may apply Theorem 6.1, the characteristic cohomology $\mathcal{C}(f_0, R)$, for any coefficient ring R, contains the subalgebra $\Lambda^* R\langle e_1, e_3, e_5, e_7\rangle$. Hence, characteristic cohomology of the complement has R rank at least 1 in all degrees between 0 and 16, except for 2 and 14, and it is rank at least 2 in degree 8.

The characteristic cohomology $\mathcal{B}^{(sy)}(f_0, \mathbf{k})$ of the link contains the subspace obtained by upper truncating the exterior algebra over $\mathbf{k}$ obtained from the same $\mathbf{k}$ vector space by removing the generator of degree 16 given by the product $e_1 \cdot e_5 \cdot e_7 \cdot e_9$. Then, we shift the resulting vector space by $2 \cdot 21 - 2 - 4^2 = 24$ to obtain 1-dimensional generators in all degrees between 24 and 39, except for 26 and 38, and it is dimension at least 2 in degree 32. We note that the Link $L(\mathcal{V}_0)$ has real dimension 39, so again a vector space generator of the characteristic subalgebra generates the top dimensional class.

7. Characteristic Cohomology for Non-square Matrix Singularities

We extend the results for $m \times m$ general matrices and matrix singularities to non-square matrices.

General $m \times p$ Matrix Singularities with $m \neq p$:
Let $M = M_{m,p}(\mathbb{C})$ denote the space of $m \times p$ complex matrices (where we will assume $m \neq p$, with neither $= 1$). We consider the case where $m > p$. The other case $m < p$ is equivalent by taking transposes. The varieties of singular $m \times p$ complex matrices, $\mathcal{D}_{m,p} \subset M_{m,p}(\mathbb{C})$, with $m \neq p$ were not considered earlier because they do not have Milnor fibers. However, the methods we applied earlier to $m \times m$ general matrices will also apply to the complement and link of $\mathcal{D}_{m,p}$. We explain that the complement has a compact homotopy model given by a Stiefel manifold. As for the case of $m \times m$ matrices, it has a Schubert decomposition using the ordered factorization by "pseudo-rotations" due to the combined work of J. H. C. Whitehead [W], C. E. Miller [Mi], and I. Yokota [Y]. The Schubert cycles give a basis for the homology and the Kronecker dual cohomology classes which can be identified with the classes computed algebraically in [MT, Thm. 3.10] (or see e.g. [D3, §8]). Thus, for appropriate coefficients, the form of both $\mathcal{C}_{\mathcal{V}}(f_0, R)$ and $\mathcal{B}_{\mathcal{V}}(f_0, \mathbf{k})$ can be given for $\mathcal{V} = \mathcal{D}_{m,p}$ and $f_0 : \mathbb{C}, 0 \to M_{m,p}(\mathbb{C}), 0$.

Then, we use the Schubert structure on the Stiefel manifolds to define vanishing compact models. This allows us to define, as for the $m \times m$ case, kite subspaces and maps to detect nonvanishing characteristic cohomology of the complement and link.

Complements of the Varieties of Singular $m \times p$ Matrices.
Let $M = M_{m,p}(\mathbb{C})$ denote the space of $m \times p$ complex matrices. The varieties of singular $m \times p$ complex matrices, $\mathcal{D}_{m,p}$, with $m \neq p$ were not considered earlier because they do not have Milnor fibers. However, the methods do apply to the complement and link as a result of work of J. H. C. Whitehead [W]. We consider the case where $m > p$. The other case $m < p$ is equivalent by taking transposes. The complement to the variety $\mathcal{D}_{m,p}$ of singular matrices and can be described as the ordered set of p independent vectors in $\mathbb{C}^m$. Then, the Gram-Schmidt procedure replaces them by an orthonormal set of p vectors in $\mathbb{C}^m$. This is the Stiefel variety $V_p(\mathbb{C}^m)$ and the Gram-Schmidt procedure provides a strong deformation retract of the complement $M \backslash \mathcal{V}_{m,p}$ onto the Stiefel variety $V_p(\mathbb{C}^m)$. Thus, the Stiefel variety is a compact model for the complement.

Schubert Decomposition for the Stiefel Variety.

The work of Whitehead [W], combined with that of C. E. Miller [Mi], and I. Yokota [Y], provides a Schubert-type cell decomposition for $V_p(\mathbb{C}^m)$ similar to that given in the $m \times m$ case. There is an action of $GL_m(\mathbb{C}) \times GL_p(\mathbb{C})$ on $M_{m,p}(\mathbb{C})$ which is appropriate for considering $\mathcal{K}_M$ equivalence of $m \times p$ complex matrix singularities. However, just for understanding the topology of the link and complement of $\mathcal{D}m, p$ it is sufficient to consider the left action of $GL_m(\mathbb{C})$ acting on M with an open orbit consisting of the matrices of rank p. As explained in [D4], the complement $M_{m.p}(\mathbb{C}) \backslash \mathcal{D}_{m,p}$ is diffeomorphic to the homogeneous space $GL_m(\mathbb{C})/GL_{m-p}(\mathbb{C})$. The diffeomorphism is induced by $GL_m(\mathbb{C}) \to M_{m,p}(\mathbb{C})$ given by $A \mapsto A \cdot \begin{pmatrix} I_p \\ 0_{m-p,p} \end{pmatrix}$. Here the subgroup $GL_{m-p}(\mathbb{C})$ represents the subgroup of elements $\begin{pmatrix} I_p & 0 \\ 0 & A \end{pmatrix}$ with $A \in GL_{m-p}(\mathbb{C})$. This gives the isotropy subgroup of the left action on $\begin{pmatrix} I_p \\ 0_{m-p,p} \end{pmatrix}$.

For successive values of m, we have the successive inclusions: $GL_{m-1}(\mathbb{C}) \hookrightarrow GL_m(\mathbb{C})$ by $A \mapsto \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}$. These induce inclusions

$$i_{m-1,p-1} : GL_{m-1}(\mathbb{C})/GL_{m-p}(\mathbb{C}) \hookrightarrow GL_m(\mathbb{C})GL_{m-p}(\mathbb{C}).$$

There is a corresponding inclusion of the spaces of matrices $M_{m-1,p-1}(\mathbb{C}) \hookrightarrow M_{m,p}(\mathbb{C})$ by $B \mapsto \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix}$. This inclusion induces a map of the complements of the varieties of singular matrices

$$\tilde{i}_{m-1,p-1} : M_{m-1,p-1}(\mathbb{C}) \backslash \mathcal{D}_{m-1,p-1} \hookrightarrow M_{m,p}(\mathbb{C}) \backslash \mathcal{D}_{m,p}.$$

The actions of the groups on the spaces of matrices commute via the inclusions of the groups with the corresponding inclusions of spaces of matrices. Thus, we have a commutative diagram of diffeomorphisms and inclusions

$$
\begin{array}{ccc}
GL_{m-1}(\mathbb{C})/GL_{m-p}(\mathbb{C}) & \xrightarrow{i_{m-1,p-1}} & GL_m(\mathbb{C})/GL_{m-p}(\mathbb{C}) \\
\simeq \downarrow & & \simeq \downarrow \\
M_{m-1,p-1}(\mathbb{C}) \backslash \mathcal{D}_{m-1,p-1} & \xrightarrow{\tilde{i}_{m-1,p-1}} & M_{m,p}(\mathbb{C}) \backslash \mathcal{D}_{m,p}
\end{array}
$$

(7.1)

The homogenenous spaces $GL_m(\mathbb{C})/GL_{m-p}(\mathbb{C})$ are homotopy equivalent to the homogenenous spaces given as the quotient of their maximal compact subgroups U_m/U_{m-p}. Via the vertical isomorphism in (4.2), the complement is diffeomorphic to the Stiefel variety $V_p(\mathbb{C}^m)$.

By results of Whitehead [W] applied in the complex case (see e.g. [D4, §3]), the Schubert cell decomposition of $V_p(\mathbb{C}^m)$ is given via ordered factorizations of matrices in U_m into products of "pseudo-rotations". For this we use the reverse flag with $\tilde{e}_j = e_{m+1-j}$ for $j = 1, \ldots, m$ and $\mathbb{C}^k$ spanned by $\{\tilde{e}_1, \ldots, \tilde{e}_k\}$. Then, any $B \in U_m$ can be uniquely written by a factorization in decreasing order.

(7.2) $$B = A_{(\theta_k, v_k)} \cdots A_{(\theta_2, v_2)} \cdot A_{(\theta_1, v_1)},$$

with $v_j \in_{\min} \mathbb{C}^{m_j}$ and $1 \le m_1 < m_2 < \cdots < m_k \le m$, and each $\theta_i \not\equiv 0 \bmod 2\pi$. Here $v_j \in_{\min} \mathbb{C}^{m_j}$ means $v_j \in \mathbb{C}^{m_j}$ but $v_j \notin \mathbb{C}^{m_j - 1}$. Also, each $A_{(\theta_j, v_j)}$ is a pseudo-rotation about $\mathbb{C} < v_j >$, which is the identity on $\mathbb{C} < v_j >^{\perp}$ and multiplies v_j

by $e^{\theta_j i}$. In [D4, §3] the results are given for increasing factorizations,; however, as explained there, the results equally well hold for decreasing factorizations. If $m_{k'} > m - p \leq m_{k'+1}$, then each $A_{(\theta_j, v_j)}$ for $j > k$ belongs to U_{m-p}. Hence, B is in the same U_{m-p}-coset as

$$B' \ = \ A_{(\theta_k, v_k)} \cdots A_{(\theta_k, v_k)}.$$

Then, the projections $p_{m,p} : U_m \to U_m/U_{m-p}$ of the Schubert cells $S_{\mathbf{m}}$ for $\mathbf{m} = (m_1, \ldots, m_k)$ with $m - p < m_1 < \cdots < m_k \leq m$ give a cell decomposition for $U_m/U_{m-p} \simeq V_p(\mathbb{C}^M)$. Furthermore, the closures $\overline{S_{\mathbf{m}}}$, which are the Schubert cycles, are "singular manifolds" which have Borel-Moore fundamental classes (see e.g. comment after [D4, Thm. 3.7]).

Cohomology of the Complement and Link.

We can give a relation between the homology classes given by the Schubert cycles resulting from the Whitehead decomposition and the cohomology with integer coefficients of the Stiefel variety, and hence the complement of the variety $\mathcal{D}_{m,p}$ (computed in [MT, Thm. 8.10a]).

Theorem 7.1. *The homology of the complement of $\mathcal{D}_{m,p}$ ($\simeq H_*(V_p(\mathbb{C}^m); \mathbb{Z})$) has for a free $\mathbb{Z}$-basis the fundamental classes of the Schubert cycles, given as images $p_{m,p *}(\overline{S_{\mathbf{m}}})$, with $\mathbf{m} = (m_1, m_2, \ldots m_k)$ for $m - p < m_1 < \cdots m_k \leq m$, as we vary over the Schubert decomposition of U_m/U_{m-p}. The Kronecker duals of these classes give the $\mathbb{Z}$-basis for the cohomology, which is given as an algebra by*

$$(7.3) \qquad H^*(M_{m,p} \backslash \mathcal{D}_{m,p}; \mathbb{Z}) \ \simeq \ \Lambda^* \mathbb{Z} \langle e_{2(m-p)+1}, e_{2(m-p)+3}, \ldots, e_{2m-1} \rangle$$

with degree of e_j equal to j.

Moreover, the Kronecker duals of the simple Schubert classes $S_{(m_1)}$ for $m - p < m_1 \leq m$ are homogeneous generators of the exterior algebra cohomology.

Proof. The computation of $H^*(V_p(\mathbb{C}^m)$ is given in [D4, Thm. 3.7]. As it is a homotopy model for the complement (7.3) follows.

Second, that the Schubert cycles form a basis for the homology follows exactly as in the proof of [D4, Thm 6.1], as does the proof that the Kronecker duals to the simple Schubert cycles provide homogeneous generators of the exterior algebra. □

Cohomology of the Link.

As a consequence of Theorem 7.1, we obtain the following conclusion for the link.

Theorem 7.2. *For the variety of singular $m \times p$ complex matrices, $\mathcal{D}_{m,p}$ (with $m > p$), the cohomology of the link is given (as a graded vector space) as the upper truncated cohomology $H^*(M_{m,p} \backslash \mathcal{V}_{m,p}, \mathbf{k})$ given in (7.3) and shifted by $p^2 - 2$.*

The Alexander duals of the Schubert cycles of nonmaximal dimension give a basis for the cohomology of the link.

Kite Spaces and Maps for $m \times p$ Matrix Singularities with $m \neq p$:

Definition 7.3. For $m \times p$ matrices with $m > p$, with $p \neq 1$ and the reverse standard flag of subspaces of $\mathbb{C}^m$, the corresponding *linear kite subspace of length ℓ* is the linear subspace of the space of matrices defined as follows: For $M_{m,p}(\mathbb{C})$, it is the linear subspace $\mathbf{K}_{m,p}(\ell)$ spanned by

$$\{E_{i,j} : r + 1 \leq i \leq m, r + 1 \leq j \leq p\} \cup \{E_{i,i} : 1 \leq i \leq r\}$$

$$
\begin{pmatrix}
* & \cdots & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & \cdots & * & 0 & \cdots & 0 \\
0 & \cdots & 0 & * & \cdots & * \\
0 & \cdots & 0 & * & \ddots & * \\
0 & \cdots & 0 & * & \cdots & *
\end{pmatrix}
$$

FIGURE 6. Illustrating the form of elements of a linear kite space of length ℓ in the space of general $m \times p$ matrices with $r = p - \ell$. The upper $r \times r$ left matrix is a diagonal matrix with arbitrary entries, and the lower right matrix is a general matrix of size $(m - r) \times (p - r)$.

where $r = p - \ell$.

Furthermore, we refer to the germ of the inclusion $i_{m,p}(\ell) : \mathbf{K}_{m,p}(\ell), 0 \to M_{m,p}(\mathbb{C}), 0$, for each of the three cases as a *linear kite map of size ℓ*. Furthermore, a germ which is $\mathcal{K}_M$ equivalent to $i_{m,p}(\ell)$ will be refered to as an *unfurled kite map of length ℓ*. We also say that a germ $f_0 : \mathbb{C}^n, 0 \to M_{m,p}(\mathbb{C}), 0$ contains a kite map of length ℓ if there is an embedding $g : \mathbf{K}_{m,p}(\ell), 0 \to \mathbb{C}^n, 0$ so that $f_0 \circ g$ is an unfurled kite map of colength ℓ.

The general form of elements, "the kites" in the linear kite subspaces have the form given in (7.4)

(7.4) $$ Q_{\ell, m-\ell} = \begin{pmatrix} D_r & 0_{m-r, p-r} \\ 0_{m-r, p} & A_{m-r, p-r} \end{pmatrix} $$

where $r = p - \ell$ and $A_{m-r, p-r}$ is an $(m - r) \times (p - r)$-matrix which denotes an arbitrary matrix in $M_{m-r, p-r}(\mathbb{C})$. Also, $0_{q,s}$ denotes a 0-matrix of size $q \times s$ and D_r, an arbitrary $r \times r$ diagonal matrix. The general element is exhibited in Figure 6.

Remark 7.4. Although the body of the kite is not square, the length ℓ denotes the rank of a generic matrix in the body, which is consistent with the square case when $m = p$. We note that to be consistent with the form of the matrices for the group representation of the complement and the Schubert decomposition for the nonsquare case, the kite is "upside down". However, elements of $\mathcal{K}_M$ allow for the composition with invertible matrices GL_m and GL_p with entries in the local ring of germs. This allows for a linear change of coordinates so the kite can be inverted to the expected form as for the $m \times m$ case.

We have an analogue of the detection result for case of $m \times m$ general matrices.

Theorem 7.5. *Let $f_0 : \mathbb{C}^n, 0 \to M_{m,p}(\mathbb{C}), 0$ define a matrix singularity. If f_0 contains a kite map of length ℓ, then the characteristic cohomology of the complement $\mathcal{C}_{m,p}(f_0, \mathbf{k})$, for a field $\mathbf{k}$ of characteristic 0, contains the exterior algebra given by*

(7.5) $$ \Lambda^* \mathbf{k} \langle e_{2(m-p)+1}, e_{2(m-p)+3}, \ldots, e_{2(m-p)+2\ell-1} \rangle $$

and each e_j has degree j.

Furthermore, the characteristic cohomology of the link $\mathcal{B}_{m,p}(f_0, \mathbf{k})$, as a graded vector space contains the graded subspace given by truncating the top degree of the exterior subalgebra (7.5) of $\mathcal{C}_{m,p}(f_0, \mathbf{k})$ and shifting by $2n - 2 - \ell \cdot (2(m - p) + \ell)$.

For the complement, $\mathbf{k}$ may be replaced by any coefficient ring R.

Proof. The line of proof follows that for the $m \times m$ general case.

Under the inclusion $i_{m-1,p-1} : V_{p-1}(\mathbb{C}^{m-1}) \hookrightarrow V_p(\mathbb{C}^m)$, the identification of the cohomology classes with Kronecker duals of the Schubert cycles implies

$$i^*_{m-1,p-1}(e_{2(m-p+j)-1}) = e_{2(m-p+j)-1} \text{ for } 1 \leq j \leq p-1 \text{ and } i^*_{m-1,p-1}(e_{2m-1}) = 0.$$

If we compose successive inclusions ℓ times to give $i_{m-\ell,p-\ell} : V_{p-\ell}(\mathbb{C}^{m-\ell}) \hookrightarrow V_p(\mathbb{C}^m)$, then the induced map on cohomology has image the algebra given in (7.5). Thus, $V_p(\mathbb{C}^m)$ provides a compact model for the complement, and the composition $i_{m-1,p-1} \circ i_{m-2,p-2} \circ \cdots \circ i_{m-r,p-r}$ with $r = p - \ell$ detects the subalgebra in (7.5).

Now using the vanishing compact model $t \cdot V_p(\mathbb{C}^m)$, we can follow the same reasoning as for the $m \times m$ case using the functoriality and invariance under $\mathcal{K}_{\mathcal{D}_{m,p}}$, and apply the Second Detection Lemma to obtain the result.

Then, as the exterior algebra satisfies Poincare duality under multiplication, we can deduce the result for $\mathcal{B}_{m,p}(f_0, \mathbf{k})$ using the same argument in the proof of [D3, Prop. 1.9] where for the shift $2n - 2 - \dim_{\mathbb{R}} K$ we replace $\dim_{\mathbb{R}} K$ by the top degree of the algebra in (7.5). This is the same as $\dim_{\mathbb{R}} V_{p-r}(\mathbb{C}^{m-r})$, which is

$$2(p-r)((m-r)-(p-r)) + (p-r)^2 \; = \; (p-r)(2(m-p)-r) \; = \; \ell(2(m-p)+\ell).$$

$\square$

Example 7.6. Consider an example of a matrix singularity which is given by $f_0 : \mathbb{C}^{12}, 0 \to M_{4,5}(\mathbb{C}), 0$ defined by the matrix in Figure 7 for which all $g_{i,j}(\mathbf{x}, 0) = 0$. For $\mathbf{y} = (y_1, y_2, y_3, y_4)$, when $\mathbf{y} = 0$, we see that f_0 contains a kite map of colength

$$\begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} & g_{1,4}(\mathbf{x}, \mathbf{y}) & g_{1,5}(\mathbf{x}, \mathbf{y}) \\ x_{2,1} & x_{2,2} & x_{2,3} & g_{2,4}(\mathbf{x}, \mathbf{y}) & g_{2,5}(\mathbf{x}, \mathbf{y}) \\ g_{3,1}(\mathbf{x}, \mathbf{y}) & g_{3,2}(\mathbf{x}, \mathbf{y}) & y_3 & x_{3,4} & y_4 \\ y_1 & y_2 & g_{4,3}(\mathbf{x}, \mathbf{y}) & g_{4,4}(\mathbf{x}, \mathbf{y}) & x_{4,5} \end{pmatrix}$$

FIGURE 7. An example of a 4×5 matrix singularity f_0, with $g_{i,j}(\mathbf{x}, 0) \equiv 0$ for each (i, j). It contains a kite map of colength 2 given when all $y_i = 0$.

2. Then, Theorem 7.5 implies that $\mathcal{C}_{\mathcal{D}_{4,5}}(f_0, \mathbb{Z})$ contains a subalgebra $\Lambda^* \mathbb{Z} \langle e_3, e_5 \rangle$. Also, by Theorem 7.5, $\mathcal{B}_{\mathcal{D}_{4,5}}(f_0, \mathbb{Z})$ contains as a subgroup the subalgebra upper truncated and then shifted by $2 \cdot 12 - 2 - (4 - 2)(2(5 - 4) + 2) = 14$. Thus, the classes $\{1, e_3, e_5\}$ are shifted by 14 to give classes in degrees $14, 17, 19$. As $\mathcal{V}_0 = f_0^{-1}(\mathcal{D}_{4,5})$ has codimension 2, the link $L(\mathcal{V}_0)$ has dimension 19 and the characteristic cohomology class in degree 19 generates the Kronecker dual to the fundamental class of $L(\mathcal{V}_0)$.

8. Cohomological Relations between Local Links via Restricted
Kite Spaces

Lastly, it is still not well understood how the structure of the strata for the varieties of singular matrices contributes to the (co)homology of the links for the various types of matrices. We use kite spaces for all of the cases to determine the relation between the cohomology of local links for strata with the cohomology of the global link. This includes as well the relation between the local links for strata with local links of strata of higher codimension. This is via the relative Gysin homomorphism defined as [D6, (1.10)], which is an analog of the Thom isomorphism theorem in these cases.

We do so by explaining how the kite subspaces provide transverse sections to the strata of the varieties of singular matrices for all three cases of $m \times m$ matrices and also for general $m \times p$ matrices. To consider all cases simultaneously, we denote the corresponding space of matrices by M and the variety of singular matrices by $\mathcal{D}_*^{(*)}$. Also, we consider the kite subspace of length ℓ of appropriate type which we denote by $\mathbf{K}_*^*(\ell)$. For the $m \times m$ cases, we also let $r = m - \ell$ (which is the same as $p - \ell$ when $m = p$).

We consider an affine subspace obtained by choosing fixed nonzero values at the entries in the tail (e.g. the value 1). When the entries in the body of the kite are 0, we obtain a matrix A of rank r and hence corank ℓ. Then, the resulting space we consider has the form $A + M'$ where M' denotes one of the spaces $M_\ell(\mathbb{C})$, $M_\ell^{(sy)}(\mathbb{C})$, $M_\ell^{(sk)}(\mathbb{C})$, or $M_{m-r,p-r}(\mathbb{C})$ which is embedded, via a map denoted by i, as the body of the kite. This provides a normal section to the stratum Σ_ℓ of matrices of corank ℓ through A. We refer to this affine subspace as a *restricted kite space*. We let $\mathcal{D}_*^{(*)'}$ denote the variety of singular matrices in M'. Then, in a sufficiently small tubular neighborhood T of Σ_ℓ we obtain $\mathcal{D}_*^{(*)} \cap T$ is diffeomorphic to $\Sigma_\ell \times (\mathcal{D}_*^{(*)'} \cap B_\varepsilon)$ for sufficiently small $\varepsilon > 0$. We refer to $L(\mathcal{D}_*^{(*)'})$ as the *local link of the stratum* Σ_ℓ.

Then i induces an inclusion $i : \mathcal{D}_*^{(*)'} \cap B_\varepsilon \subset \mathcal{D}_*^{(*)}$. There is the induced map i^* on cohomology which sends the exterior algebra giving the cohomology of $M \backslash \mathcal{D}_*^{(*)}$ to the algebra (7.5). This is a consequence of the proofs of Theorems 6.1 and 7.5. Using this we have consistent monomial bases for the cohomology of the complement. This allows us to define consistent Kronecker pairings giving a well-defined relative Gysin homomorphism (as defined in [D6, (1.10)]). There is the following relation between the cohomology of the local link $L(\mathcal{D}_*^{(*)'})$ and the link $L(\mathcal{D}_*^{(*)})$.

Corollary 8.1. *The relative Gysin homomorphism*

$$i_* : H^*(L(\mathcal{D}_*^{(*)'}); \mathbf{k}) \to H^{*+q}(L(\mathcal{D}_*^{(*)}); \mathbf{k})$$

increases degree by $q = \dim_\mathbb{R} M - \dim_\mathbb{R} M'$, which in the various cases equals for the $m \times m$ cases: $2(m^2 - \ell^2)$ for the general matrices; $(m-\ell)(m+\ell+1)$ for symmetric matrices, $(m-\ell)(m+\ell-1)$ for skew-symmetric matrices (with m and ℓ even); and for $m \times p$ matrices $2(p^2 - \ell^2)$.

It is injective and sends the Alexander dual of the Kronecker dual of a class corresponding to a monomial in the algebra (7.5) to the corresponding Alexander dual of the Kronecker dual of the image of that class as an element of the cohomology of the complement $M \backslash \mathcal{D}_*^{(*)}$.

Proof. By the above remarks, there is defined the relative Gysin homomorphism. If i denotes the inclusion of the reduced kite space into the space of matrices, then the induced map on cohomology of the complements, denoted i^*, (with coefficients $\mathbf{k}$ a field of characteristic 0) is surjective. We use the identification of the monomials $e_{\mathbf{m}}$ with the Kronecker duals denoted $e_{\mathbf{m}}^*$. Then, the inclusion i_* is the dual of i^*. Thus, the dual homomorphism for homology i_* is injective. When this is composed with Alexander isomorphisms (via the Kronecker pairings), it remains injective. By the properties of the corresponding cohomology classes of the links resulting from applying Alexander duality have the effect of raising degree by the difference $\dim_{\mathbb{R}} M - \dim_{\mathbb{R}} M'$ for each of the four types. These are then computed to give the stated degree shifts. $\qquad\square$

We also mention that there is an analogous version of this corollary for the case of the local link for a stratum $\Sigma_{\ell'}$ included in the local link of a stratum Σ_{ℓ} for $\ell' < \ell$. As an example we consider

Example 8.2. For the stratum $\widetilde{\Sigma_2} \subset Sym_5(\mathbb{C})$, the local link has reduced cohomology group isomorphic to $\widetilde{\Lambda * \mathbf{k}}\langle e_1, e_5 \rangle [4]$. However, the effect of Alexander duality on elements does not correspond to a shift. The reduced cohomology of the local link complement is spanned by the generators $\{e_1, e_5, e_1 e_5\}$ with Kronecker duals denoted $\{e_{1*}, e_{5*}, (e_1 e_5)_*\}$. Then, the corresponding Alexander dual generators for the reduced cohomology of the local link, denoted $\{\widetilde{e_1}, \widetilde{e_5}, \widetilde{e_1 e_5}\}$, have degrees in cohomology $9, 5, 4$ in that order. Note under the shift representation $\widetilde{e_1 e_5}$ corresponds to the shift of 1.

Also, the link $L(\mathcal{D}_5^{(sy)})$ has cohomology group $\widetilde{\Lambda * \mathbf{k}}\langle e_1, e_5, e_9 \rangle [13]$. Then, as i^* is surjective, i_* maps the Kronecker duals $\{e_{1*}, e_{5*}, (e_1 e_5)_*\}$ to homology classes of the same degrees for the complement $Sym_5(\mathbb{C}) \backslash \mathcal{D}_5^{(sy)}$. Then, these elements correspond under Alexander duality to cohomology classes $\{\widetilde{e_1}', \widetilde{e_5}', \widetilde{(e_1 e_5)}'\}$ for the link $L(\mathcal{D}_5^{(sy)})$, having degrees $27, 23, 22$. We see that the increase in degree is $2(15 - 6) = 18$ as asserted for the relative Gysin homomorphism..

However, one key point to note is that for the cohomology group of the link represented by the truncated and shifted exterior algebras, the relative Gysin homomorphism does not map the shifted classes to the corresponding shifted classes. For example, for the local link of $\Sigma_2 \subset Sym_5(\mathbb{C})$, e_1 corresponds to $\tilde{e}_1$ which maps to a cohomology class of degree 27, while for the link $L(\mathcal{D}_5^{(sy)})$, e_1 corresponds via the shift representation to a cohomology class in the link of degree 14.

Remark 8.3. For $m \times p$ finitely $\mathcal{K}_M$-determined matrix singularities $f_0 : \mathbb{C}^n, 0 \to M_{m,p}(\mathbb{C}), 0$, if $n < |2(m - p + 2)|$, then by transversality, $\mathcal{V}_0$ has an isolated singularity and so a stabilization provides a Milnor fiber as a particular smoothing. As yet there does not appear to be a mechanism for showing how this Milnor fiber inherits topology from $M_{m,p}(\mathbb{C}) \backslash \mathcal{D}_{m,p}$. However, for $(m, p) = (3, 2)$, Frühbis-Krüger and Zach [F], [Z], [FZ] have shown that for the resulting Cohen-Macaulay 3-fold singularities in $\mathbb{C}^5$, the Milnor fiber has Betti number $b_2 = 1$, allowing the formula of Damon-Pike [DP, §8] to yield an algebraic formula for b_3. It remains to be understood how this extends to larger (m, p).

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